In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institution shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

THE DEVELOPMENT OF WORK SAMPLING PROCEDURES BASED ON SURVEY SAMPLING THEORIES AND PRACTICES

A THESIS

Presented to

The Faculty of the Graduate Division

By

Vernon Eugene McBryde

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

In the School of Industrial Engineering

Georgia Institute of Technology

August, 1963

12R
5L

THE DEVELOPMENT OF WORK SAMPLING PROCEDURES

BASED ON SURVEY SAMPLING THEORIES AND PRACTICES

Approved:

Date approved by Chairman: April 6, 1964

FOREWORD

This research resulted from an awareness that work sampling practitioners are not making full use of many refined sampling techniques which have evolved since work sampling was established as a practical management analysis technique. The author's participation in the National Science Foundation Summer Institutes for College Teachers of Statistics at the University of Wyoming in 1959 and at Iowa State University in 1961 made him aware of the extensive field of survey sampling and served to raise the question of the applicability of some of the theories in that field to work sampling. Dr. Joseph J. Moder, who guided this research, sustained this interest and provided encouragement which led to this study. His assistance in formulating the scope of the study and in interpreting many of the advanced concepts of sampling methodology has been invaluable.

Many helpful suggestions were also given by Dr. Harold E. Smalley and Dr. James W. Walker, members of the research committee. These are gratefully acknowledged. Special thanks are due Professor Lynwood A. Johnson for valuable computer assistance during the course of the research, and Professor Jackson H. Birdsong for his continued interest and support.

My graduate study has been with the aid of a Ford Foundation Fellowship. Without such assistance this work would have been most difficult. In conjunction with this assistance, Professor Frank F. Groseclose has been instrumental in providing an atmosphere conducive to graduate study. His assistance throughout the tenure of the fellowship is greatly appreciated.

Finally, the encouragement and understanding of my wife Nell, daughter Pamella, and son Bryan, are affectionately acknowledged. To them I dedicate this work.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

The general objective of this study was to develop new random sampling methodologies for use in work sampling. These methodologies are based on certain theories and practices which have already been tested by practitioners in survey sampling. Specific objectives were (1) to explore the limitations of simple random sampling, the conventional approach in work sampling, (2) to develop a work sampling model which employs the concept of stratification for designing more efficient sampling methods, (3) to devise a means for accurately treating work sampling data which are collected in clusters, and (4) to present methods for the efficient design of work samples which cover activities that are too scattered, geographically, to allow the use of simple methods.

Prior to considering the objectives enumerated above, the study presents a comprehensive review of work sampling history, commencing with its inception by L. H. C. Tippett of the British textile industry, and terminating with the most recent developments in the field. The review is oriented toward literature which deals with theoretical rather than applied aspects of the tool.

In the analysis of simple random sampling, the conventional approach to work sampling, a general pattern is set for the entire study. A discussion pertaining to what the sampling unit should be in work sampling results in the choice of an "instant" of time as the basic unit, but selected in a manner which is administratively convenient and theoretically acceptable. The general terminology and notation for the entire study is

introduced with respect to the simple random work sampling model, and graphical means for designing simple random work samples for stated levels of precision are presented.

An investigation of the variation of the sample variance for a simple random estimator results in a graph which allows the practitioner to ascertain the error involved in estimating the true variance by using the sample variance. A cost model is developed for the simple random design which encompasses the components: (1) fixed costs, (2) sampling costs, and (3) "loss due to error" costs. A method for designing the sample to minimize total costs (or to achieve a stated cost) is compared with the method of designing for a stated precision, and the resulting analysis illustrates the approach to use in the final selection of a sample size.

The simple random work sampling model, and the assumptions pertaining thereto, are demonstrated and tested by simulated sampling of a hypothetical activity introduced for that purpose. Results of the simulation are given and compared with the theoretical predictions to show that the estimators may be approximated by normal distributions, and that estimator means and variances are adequately described by the simple random model.

The concept of stratification is introduced by citing some common population structures in work sampling which are amenable to stratification. General estimators and their variances are presented for stratified random work sampling and investigations of optimal allocations of the sample to the strata are presented. After considering allocations to minimize estimator variance, and allocations to minimize total sampling costs, subject to stated sample sizes, it is concluded that the self-weighting feature and the simplicity of proportional allocation make this method more conducive to work sampling uses. The nature of gains in precision of proportional stratified random work sampling over simple random work sampling is shown quantitatively in terms of the variance between strata. The gain is also illustrated by designing a sample to yield an estimator with the same precision, but requiring fewer observations than the simple random estimator.

The stratified model is also illustrated by simulated sampling of the same hypothetical activity previously introduced. Comparisons are made of the results from the stratified samples and those from the simple random samples.

A general cost model is developed for the stratified estimator and design considerations for minimizing total costs or for designing to a stated cost are given. The necessity of balancing the sample design in terms of costs and precision is discussed and illustrated.

An analysis of the assumption of independence between work sampling observations made simultaneously in groups indicates the need for a work sampling model which is free of this assumption. The techniques of cluster sampling in survey practices are drawn upon for developing models for cluster work sampling when the populations either are or are not stratified. The pattern set by the development of the other models is followed in developing estimators, their variances, and design characteristics in terms of both precision and costs. A model for the case of simple sampling of clusters is presented for both variable and constant cluster sizes. Investigation shows that the variation in sizes of clusters in work sampling is usually small enough to permit the use of constant cluster size theory.

Variance comparisons are made between cluster sampling and simple random sampling; however, the basic gains in the use of cluster sampling over simple random sampling are shown to be in the form of costs. Cost models for both simple random and stratified random sampling of the clusters are developed and total costs compared between these and simple random work sampling costs.

Computer simulation affords a means of illustrating the cluster models in the same manner that was used in the previous cases. The results of the simulations are presented along with those predicted by the theory to show their close agreement.

The final model presented in the study is for the analysis of large complex activities, situated perhaps at several geographic locations. These activities are presented as population structures which are made up of several levels or stages such as factories, departments within factories, work centers within departments, etc. The general method of sampling which is presented in this case is called Multi-Stage Work Sampling. The model actually developed in terms of stages is found to be of little practical use in most work sampling situations, and is thus used to introduce more realistic means of sampling. The major disadvantage of the general model is that it requires equal size units at each stage of sampling. In addition, the estimator variances become prohibitively difficult to write when several stages of sampling are used with different sampling schemes at each level.

The methods developed for circumventing the problems associated with the general Multi-Stage model are: (1) sampling with probabilities proportional to size and (2) sampling with equal probabilities from "paper zones" which allow the creation of units of equal size. In the first case, the population structure is kept intact and units are selected for the sample with probabilities proportional to their sizes. In the latter case, the population is restructured in such a way that "paper zones" of equal size are formed and subsequently sampled in a simple fashion. Estimators and their variances are developed for each of these procedures.

The study is concluded with the presentation of a procedure for choosing the proper work sample design in a given situation. This results in a need for the practitioner to be reasonably familiar with the activity being sampled in order to be able to provide the answers to a series of simple questions pertaining to the population structure.

CHAPTER I

INTRODUCTION

Purposes

The primary purpose of this study is to develop more efficient methods of conducting work sampling studies using sampling techniques which have been used successfully in the field of survey sampling. This study will show that knowledge of the population structure under study may be exploited in designing the sampling method, resulting in a more efficient practice as measured by the accuracy of the results and the overall cost of the study. Relevant knowledge of the population, required for a more efficient sampling design, usually is not difficult to obtain. That individual workers show a marked difference in performance, that certain days are more susceptible to work stoppages than others, and that workers perform their duties in teams are examples of the types of information required.

Study results will appear in the form of sampling models, categorized as follows:

(1) Simple Random Work Sampling

(2) Stratified Random Work Sampling

(3) Cluster Work Sampling

(4) Multi-Stage Work Sampling.

Each of the first three models represents a unique way of sampling whereas the fourth model is a method of sampling which may employ one or more of the first three models applied in various ways at several stages of sampling.

The gains over conventional methods which one may achieve by employing these categorized work sampling models will be depicted in terms of both the precision* of the estimators and the overall costs of the study. Illustrative examples, portraying both the sampling scheme and the relative economic gains of each model, will be incorporated. Comparisons will be made between each model and the conventional method of simple random work sampling. Finally, decision rules for determining the best method of work sampling in specific situations will be specified.

The Nature and Importance of Work Sampling

The extensive use of work sampling data in managerial activities, as evidenced by the literature survey included herein, appears to warrant a quest for more efficient sampling methods. There is reason to believe that work sampling methods can be improved by a study of their theoretical foundation.

The measurement of work is an important part of the industrial engineer's duties, and considerable attention has been given this topic since F. W. Taylor introduced the concepts of work study in 1888. Work sampling has evolved from efforts to improve methods of measuring the time which workers devote to various elements of work activities.** Attempts to ascertain the amount of idle time in a job resulted in the first major use of sampling methods in work studies; these are commonly referred to as "ratio-delay" studies (39). Extensions of ratio-delay

---

*The term "precision," as used in this study, refers to the variability of an estimator, and in the case of unbiased estimators, may be regarded as synonymous with accuracy.

**The word "activity" is used to describe the engagement of one or more subjects in a designated action of any type.

techniques to the study of indirect work, utilization studies of equipment and men, and the measurement of relatively non-repetitive work have resulted in a tool which is frequently utilized by industrial engineers. This tool is better described by the broader term "work sampling."

The fact that work sampling can be applied for measuring a broad range of activities makes it a widely used method of analysis. This is especially true in the current technological age in which industrial engineering analyses are being directed towards a total systems approach. Work sampling is of importance in the fact finding stage of managerial decision making processes, and proper analysis of the data using statistical methodology is profitable in providing rational, reliable decisions. It, as well as other statistical sampling techniques, becomes more important as the scope of analysis enlarges. In this respect, improvements in the analytical machinery used acquire greater significance.

The most common procedure for sampling on activity utilizes theory which requires randomly-spaced observations.* For example, if a machining operation is studied for the purpose of determining the proportion of total time spent in each of three categories, say (1) machining, (2) load and unload, and (3) non-productive, a series of n observations of the process is made at random and the state of the process is recorded each time it is observed. The ratios n_1/n, n_2/n, and n_3/n (referred to later as p_1, p_2, and p_3) represent unbiased estimates of the true

*The use of random observations erases many theoretical complications since probability theory can only be applied in analyses when observations have been made at random. While it is sometimes advantageous from a practical point of view to use some other manner of selecting observations, this study will, in general, deal with random sampling. A situation which represents the most common logical departure from randomness is systematic sampling, which involves only a random starting point.

proportions of time spent in each category (represented by P_1, P_2, and P_3), where n_i represents the number of the n observations at which the activity was found to be in the ith state.

The typical approach is to make preliminary estimates of P_1, P_2, and P_3 and use these in determining the total number of observations, n, which will be needed to provide an estimate, p_i, which will be within a stated distance d of the true value, P_i. This determination is made by using the fact that the random variable p_i is approximately binomially distributed and has a variance equal to

$$\sigma^2_{p_i} = \left\{\frac{P_i(1 - P_i)}{n}\right\} \left\{\frac{N - n}{N - 1}\right\} \doteq \frac{P_i(1 - P_i)}{n} , \tag{1}$$

in which N is the total number of possible observations. If the desired level of reliability is $1 - \alpha$, then this sampling requirement becomes

$$P \left\{ - d \leq p_i - P_i \leq d \right\} = 1 - \alpha . \tag{2}$$

Now, if n is sufficiently large and not a significant portion of N (which presents no problem in work sampling since N is assumed to be very large), then one may use the normal approximation to the binomial distribution and may express equation (2) as:

$$P \left\{ - \frac{d}{\sigma_{p_i}} \leq \frac{p_i - P_i}{\sigma_{p_i}} \leq \frac{d}{\sigma_{p_i}} \right\} = 1 - \alpha . \tag{3}$$

The term $\frac{p_i - P_i}{\sigma_{p_i}}$ is approximately a standard normal variable with a mean equal to zero and a variance equal to one. Upon substituting the upper $\alpha/2$ percentage point* of the standard normal distribution, $K\alpha/2$, into equation (3), the following expression is apparent:

$$\frac{d}{\sigma_{p_i}} = K_{\alpha/2} = \frac{d\sqrt{n}}{\sqrt{P_i(1 - P_i)}} \quad , \tag{4}$$

from which

$$n \doteq \frac{(K_{\alpha/2})^2 P_i(1 - P_i)}{d^2} \quad . \tag{5}$$

Using this expression and the preliminary estimates of the P_i's, the total sample size, n, for a stated value of d, may be determined. The usual procedure is to calculate a value of n for each P_i and then choose the one which is largest. This practice results in a higher reliability than that specified for all the other P_i's. The choice of d in this analysis is an economic consideration and is a function of the "loss due to error" in the estimates, a component of cost which will be considered in more detail later. It should be mentioned at this point because a value of d is often chosen arbitrarily and without careful regard to its economic consequences.

The value of α reflects the level of confidence the decision maker desires to have in achieving the stated precision; e.g., he may wish the estimate, p, to be within 0.02 of the true proportion, P, with a probability of 0.99, in which case α is 0.01. This means that should the estimation process be repeated a large number of times, the estimate would be expected to fall farther than 0.02 from P only one

*The $\alpha/2$ percentage point of the normal distribution is that point where $\alpha/2$ of the area under the curve falls to the right of it.

per cent of the time.

Improvements in Work Sample Design

When the foregoing procedure is followed, the stated levels of reliability in the estimates (reflected by the assumed values of d and α) are fairly certain of being obtained, the uncertainty being that association with the approximation of the binomial distribution. However this reliability is often obtained at the expense of taking large samples. While the collection of data is not the only contribution to expense in work sampling studies, it usually represents the major portion; hence large samples mean costly studies. A sampling scheme which would attain the same precision as some other scheme, but with a smaller sample, would thus have the potential of being a more desirable scheme in terms of cost. The components of costs associated with each of the sampling schemes presented later in this study will be enumerated and investigated. It may be observed, that in general, too large a sample results in a waste of resources, whereas too small a sample diminishes the utility of the results. It is the proper balance of these two factors which results in the "correct" or most efficient sample size.

Sampling practices have been developed in the field of survey sampling which allow the use of known population characteristics in designing the sample to achieve greater efficiency. In the study which follows, an analysis of work sampling problems in the light of survey sampling techniques is presented for the purpose of improving the sampling procedure in work sampling studies and thereby increasing the overall efficiency of such studies. Efficiency will be measured in terms of costs and/or precision of the estimators. The various models which are presented are developed in such a way that the costs associated with each procedure are an integral part of the model. The nature of the improvements which each sampling model affords is indicated in the following synopses.

Simple Random Work Sampling

The work sampling procedure illustrated earlier in this chapter will be referred to as Simple Random Work Sampling and will be discussed more fully in Chapter III. This method of sampling represents one of the four basic models to be presented. Each of the basic models will reflect the primary objective of work sampling, which is to obtain reliable estimates of the P_i's, the fraction of time spent performing the ith element of an activity over a time period T.* T is the period of the study and must be representative of future periods of activity if the estimates are to be used for predictive purposes. The nature of this requirement is analyzed and explained more fully in Chapter VII.

If T, the span of the study, is measured in some finite unit of time, the population being sampled may be conceived as consisting of a finite number of T units. The estimate of P_i given earlier $(p_i = n_i/n)$ is unbiased and it has a variance which may be calculated by use of equation (1) if the population being sampled can be regarded as consisting of an infinite number of units. Chapter III will be devoted to a detailed

*Survey sampling generally involves the estimation of means, totals, and/or percentages pertinent to population characteristics. Since work sampling deals with estimating the proportions of time spent performing various elements of an activity, the estimation of this type of characteristic will be of major importance in this research. If certain other data are collected with respect to the activity being analyzed, such as total time elapsed, total units produced, etc., estimates of total or average time per unit can be made for the total activity as well as for its individual elements. These are often of interest for use in setting standards of performance.

analysis of this facet, and others, of this method of sampling and will present an enumeration of its advantages and limitations.

The simple random model will be used for introducing the general notation for the entire study. In addition, cost analyses in work sampling may best be illustrated by their application to the simple random model. An investigation of the opposing design requirements of minimum cost and stated levels of precision will be introduced with the simple random model and will be expanded later to encompass the more complex sample designs.

Graphical means will be provided for determining the sample size and the adequacy of the sample estimate of variance. A hypothetical activity will be introduced to simulate the results provided by the estimators in simple random work sampling. These will be used in illustrating the gains in efficiency provided by other sampling schemes.

Stratified Random Work Sampling

If the time period T can be divided into L periods between which the fractions of time spent performing the various elements are significantly different, more efficient estimates of the P_i's can be obtained by taking a specified number of observations within each of the L periods (strata), estimating the P_i's in each stratum, and combining these L estimates.* For example, if the element of interest (state of the activity) in a study appeared predominately on Mondays and Fridays, then the allocation of the total sample would be made in such a way as to take advantage of this additional information about the population.

---

*It may also be desirable to obtain estimates on certain sub-populations, in which case these sub-populations should be separated as strata and sampled independently.

Or, if the subjects involved in a given activity are significantly different in the amounts of time they need to perform the various elements of an activity, the sample may be stratified by individual subject. This approach utilizes the concept of stratification and allows the use of estimators which reflect only the variation within strata. A method for incorporating these procedures in work sampling practice is developed in Chapter IV in terms of a Stratified Work Sampling model.

Stratified estimators and their variances are developed and are illustrated by designing a stratified sample to estimate the parameters of the same hypothetical activity introduced in the previous chapter. An investigation of various methods of allocating the sample to the chosen strata will be made, and a single best method will be sought. The nature of the gains in precision and cost afforded by stratification will be discussed and subsequently substantiated by simulated results.

Cluster Work Sampling

Where several subjects (say S) are engaged in the activity being studied, it is common practice to take randomly-spaced observations as above, to observe all members in the group of subjects simultaneously, and to record the activity of each of the S subjects. Analyses are generally made under the assumption that all S observations obtained at a single point in time are independent and can thus be analyzed by use of the binomial theory. It is obvious that many activities so studied do not yield sets of S independent observations. For this reason, the assumption usually results in an underestimate of the variance of p_i; consequently, the specified levels of reliability are incorrect. In case the S subjects are not working independently of each other, a measure of the degree of association between them can be determined from the data; this can be used to provide accurate estimates of the variance of p_i.

The primary purpose of cluster sampling is not to minimize the total number of observations, but rather to gain the most reliable sample per unit of costs. When the work situation contains more than one component, e.g., when workers are performing as groups, or when machines are arranged in banks, the cost of observing the entire group is little more than the cost of observing a single member of the group. Hence, to record the additional information which is available (which is usually done) is a natural improvement over simple random work sampling methods.

The development of a work sampling model which is based on the assumption of sampling groups of subjects rather than a single subject forms the basis for the third specified model which will be presented in Chapter V. The model for Cluster Work Sampling will be developed along the same lines used for the previous models. After a general discussion of the nature of clustering in work sampling, estimators and their variances will be presented for designing this type of work study. Models which employ simple as well as stratified sampling of the clusters will be developed. The merits of these models will be illustrated in terms of variance and cost comparisons with the conventional simple random model. As before, the model will be demonstrated by simulated studies of the hypothetical population which is presented for that purpose.

Multi-Stage Work Sampling

The foregoing work sampling methods are suitable for studies which are concerned with single activities usually found in one geographic location. Settings in which activities occur within a single department or area are typical. However, a problem arises when the activity is performed in multiple areas, departments, or even factories. Management often requires data which cover multiple operations, and in this case the sampling methods presented previously become prohibitively expensive. Probability sampling methods make it possible to obtain estimates from such structures of activity by sampling at different levels (gross to minute) of the population. For example, a sample of departments may be chosen, then a sample of work areas within departments, etc., until ultimately a random sample is obtained on the activity of interest. By applying the proper methods of analysis, reliable time estimates for the whole population may be made from the resulting data.

The methods of multi-stage sampling, as practiced by survey sampling practitioners, afford a means for developing such a model in work sampling. This method will be presented in Chapter VII as a general multi-stage work sampling model; however, due to severe limitations on the units to be sampled at the various stages, the multi-stage model, in its general form, is not expected to be representative of practical situations in work sampling. To circumvent this problem, methods for sampling variable size units will be presented which do not follow the general model. While the methods which will be developed will not be expected to have wide application in work sampling, they will provide a means for efficient handling of the occasional large scale activity survey.

Scope of Study

The foregoing synopses indicate that sampling is an art which requires more than a superficial knowledge of the underlying theories.

It requires intelligent combinations of judgment and theory, combinations that allow the practitioner to design sampling plans which become more nearly optimal. Considerations of cost and precision are regarded as the most important factors in sample design, and these are treated independently as well as simultaneously for each of the sampling models advanced in this study.

It is unlikely that a set of rules could be established which would accommodate all possible sampling schemes which one might employ in work sampling situations. It is therefore not the intent of this research to provide a stereotyped approach to work sampling which could be applied routinely without knowledge of the population under study. Rather, it is hoped that this study will lead to results which will reduce the amount of statistical theory required by the work sampling practitioner in designing an efficient sampling plan. Unique approaches to sampling can be devised to fit any situation if one is sufficiently familiar with the assumptions and limitations of sampling. Therefore the scope of this study is such that it includes those activity structures which one is most likely to encounter in a work sampling study. Deviations from the models presented are to be expected in actual practice.

Limitations of Study

This research is restricted to the study of selected, random sampling schemes for use in work sampling; it is not concerned with sampling versus other methods of obtaining work study data. Analytical treatments are general as presented in final form; however, illustrations are included which show the nature of specific applications in each case.

Measures of effectiveness are presented, and each model is evaluated in relation to simple random work sampling. This particular model is chosen as a base for comparison because, in addition to being the simplest of the sampling models, it is the most common sampling approach in conventional work sampling practice. The measures of effectiveness employed are the precision of estimators and overall costs.

The mathematical derivation of sampling theorems is avoided whenever adequate references are available. Some of the more complicated mathematics of sampling theory is excluded in order to provide a sampling methodology which will be easy to understand by current work sampling practitioners.

Objectives

The considerations related in the present chapter give rise to the objectives of this research. The general objective of the study is to develop a new theoretical basis for work sampling in the form of categorized sampling models, incorporating provisions for assessing the nature of the population being sampled, and hence the specification of the proper theoretical model which will accommodate cost and reliability constraints. Specific objectives are:

(1) To explore the limitations of simple random sampling in work studies,

(2) To employ the concept of stratification for designing more efficient work sampling studies,

(3) To develop a work sampling model which adequately treats observations made in clusters,

(4) To present an approach for work sampling applications which pertains to broad, complex activities.

An important design factor which has heretofore been largely ignored in work sampling practice is the impact of costs. This phenomenon will be carefully considered with regard to each of the above objectives and methods for designing a work sampling study to minimize costs or to achieve a stated cost will be presented for each case.

CHAPTER II

REVIEW OF THE LITERATURE

A Note on Survey Sampling Practices

Man has undoubtedly used sampling in an informal fashion since prehistoric times. Most human actions are based on incomplete information because human nature prompts one to act when he feels that he has "enough" information to make a correct response. The information available in a given case is seldom more than a sample of what might be obtained. The responses of the senses -- smelling, tasting, feeling -- are nothing more than acts of sampling. Opinions which are formulated and decisions which are made are in general based on limited observation. The sampling process can be looked upon as man's way of obtaining more information about the world in which he resides.

Formalized approaches to sampling are not so old. Although some of the mathematical tools which make sampling a formal method of data collection were being developed as early as the seventeenth century by Bernoulli and others, current methods of sampling have been developed primarily since the beginning of the twentieth century. In the interim, man continued to sample his environment informally and drew conclusions which were extensions of his limited observations, seasoned by his judgment.

As Stephan points out in his survey of sampling history (47), it would be an impossible task to ferret out the many practices and activities which finally led to formalized methods of sampling. The earliest account of formalized sampling appear in studies of mortality rates and agricultural yields. Halley (28) took the mortality statistics of one small town to determine a life table for man in 1693. Studies by Sir John Lawes as early as 1852 (33) show that wheat yields for England and Wales were estimated from annual records kept at Rothamsted. Other early accounts of sampling applications involve the measurement of such things as employment, wage rates, and prices. Stephan presents brief chronological reviews of the development of sampling procedures in four broad areas:

(1) Agricultural crop estimates

(2) Economic statistics of prices, wages, etc.

(3) Statistical phases of social surveys and health studies

(4) Public opinion polling.

These reviews deal primarily with the early, crude methods of sampling, the theoretical level of which is adequately depicted by Stephan as follows:

> While, in these early instances, the sampling procedures were simple and usually employed uncritically with no great attention to accuracy and representativeness, it should be noted that the problems of observing and recording data were almost always far more serious than the problems of sampling. Modern sampling procedures, had they been available and had they been applied effectively, would have permitted the use of smaller samples and made possible in most cases more careful fieldwork. Through these developments and those in other fields of statistical study, there arose a great opportunity for the use of better sampling procedures that has been exploited only in part up to the present time.

This quote, which appeared in 1947, indicates the relatively recent growth of sampling methodology. The adaptation of a number of the recent findings in survey sampling to the methodology of activity sampling is the major aim of this research.

The characteristic which differentiates modern methods of sampling from earlier methods lies in the use of probability theory. The modern approach is to explicitly define the units which make up a given population and then to select units from the population at random using various schemes of selection, but always with the knowledge of the probability with which a given unit will be chosen. The use of this approach has become prominent in the twentieth century and attains its significance from the fact that estimates made from such samples possess determinable characteristics, namely accuracy and precision (reliability). These determinations are possible only because of the probabilistic measures which are available. In spite of successful applications of the probabilitistic approach in the early 1900's, it made very slow inroads into the then existing statistical methods. This fact is supported by the following quote from Stephan (47):

> At the Rome Session of the International Statistical Institute in 1925 a resolution was adopted recommending the use of sampling for statistical purposes with appropriate precautions as to its representativeness, mathematical statement of the precision, and full description of the methods employed. The reports that were submitted by the commission that drafted the resolution presented evidence of the usefulness of sound sampling procedures but both the reports and the resolution had less immediate effect on statistical practices than might have been expected. Eight years later, in 1934, Neyman revived the discussion of these reports in a notable paper before the Royal Statistical Society.

Widespread practice of the approach indicated above was not accomplished until the 1930's. From 1933 to 1940, various New Deal programs such as FERA, CWA, WPA, etc., increased the U. S. government's need for statistical work and hence for sampling. The enlistment of leading statisticians in the reorganization of governmental statistical work focused attention on the need for further improvement of the sampling theories then in use (27). Governmental activity has stimulated much of the research which has brought about the current state of the art. The literature (47) reveals several statistical surveys which were of national importance and in which sampling techniques were utilized. In many of the cases, the approach to sampling was not the most efficient in terms of current knowledge, however it was through experience gained in such studies that progress in the development of a theory was made.

These early approaches to analyses which involved probabilistic concepts centered around methods of simple random and systematic sampling. However the findings of experimental studies and the concentration of attention on the problem of efficient data collection caused sampling practice to evolve into complex systems which were designed to fit the population structures being studied.

The methods currently used in sampling reflect the accumulation of knowledge over many years of practice in a number of fields. The flexibility of modern theory results from many applications and experiments which have been made in the past for the specific purpose of improving the theory. The intelligent employment of these methods is to seek a combination of theory and practice which will fulfill requirements of economy, convenience, and accuracy in whatever situation arises. Current sampling practices encompass many different methods, some of which are as follows:

(a) The technique of selecting items at random,

(b) The determination of the kind of sampling unit that will have the most desirable properties,

(c) Subdivision or stratification of the population, in an advantageous manner,

(d) The use of variable sampling proportions,

(e) Subsampling and multistage sampling in which the sample itself is sampled in turn,

(f) Drawing two or more units from each ultimate subdivision to permit estimation of the sampling error from their differences,

(g) Use of information provided by variables whose values are known that are correlated with the one that is being studied, and many other procedures.

The contributions to this body of knowledge from the field of agriculture have been among the most noteworthy of any single profession. R. A. Fisher's approaches to statistical analysis and experimentation have contributed more than any other single individual's efforts to the development of effective systems of experimentation in general (23). His statistical concepts form the basis for current methods of data analysis, an important segment of sampling practice. Fisher's major contributions came during and after the 1920's.

Mass production brought into existence another fertile field for the application of statistical methods. Western Electric engineers were the first to apply the theory of probability and statistics to the control of quality as well as to inspection systems. This was accomplished in 1923 (46). The objective of these inspection procedures was to design a sampling procedure which would cut the costs of inspection but maintain control of the quality of the product. Methods which were designed were formulated to fit the conditions of manufacture and the required specifications in a given situation. Early contributors to the development of these methods were Shewhart, Dodge, Moline, Fry, and others (46). The spread of these concepts and techniques, both in industry and in agriculture, was facilitated by the founding of the Royal Statistical Society in 1933 and the subsequent publications which it supported.

The continuing demands for large scale surveys of various economic and social facts resulted in another field of endeavor (survey sampling) which has also contributed greatly to the improvement of sampling procedures. The many concepts, devices, methods, and techniques which have evolved in this area are currently well documented in such references as (15, 20, 31).

The History of Work Sampling

The brief survey of the general history of sampling presented in the foregoing section emphasizes the large scale applications of sampling. The extension of sampling procedures as aids in the analysis of smaller populations is reflected in the rapid development of statistical procedures in a number of professions in recent years. One such profession is that of engineering, especially in the field of Industrial Engineering. Statistical tools are rapidly becoming the means by which the industrial engineer replaces less scientific qualitative analyses of systems and their components with more scientific, quantitative analyses.

Aside from the work in inspection and quality control referred to earlier, one of the first industrial engineers to recognize the potential of statistical analyses in industry was L. H. C. Tippett of the British Textile Industry. His duties included the analysis of weaving operations which eventually led to a search for a method of work measurement which would short circuit the tedious method of studying hundreds of looms in a single building using stopwatch time study. His research resulted in the development of work sampling, the basic subject of this research. The following quote from Tippett's original paper in 1934 indicates the initial ideas which led to the development of the tool (48).

> The work was tedious; and as it was practical to record only two or three or four looms at a time, I had to move about the shed and observe many looms in turn before a reasonable reliable average could be determined ... I was unable at the time to use all of the detail of this information and was on the lookout for a method of observation that would be less laborious and wearisome, even if somewhat less informative. One day a weaving manager remarked, "I can tell at a glance whether the weaving in the shed is good. If most of the weavers are bent over their looms mending warp breaks, weaving is bad; if the weavers are mostly watching running looms, weaving is good." It instantly became clear that a snapshot of the state of the looms in a shed taken at any instant was in some way an indication of the rate of production in a short interval surrounding that instant and of the losses in output due to various causes.

The procedure developed by Tippett and referred to as "snap-reading" was simply a process of sampling in which the analyst made observations of the activity at preselected times and recorded the state of the activity (See Chapter I, pp. 3-5). Tippett made extensive studies (49) to justify the use of binomial theory in analyzing the data obtained from snap-readings. His comprehensive analysis of variance tests showed that for many cases the residual variance was predictable from the binomial law. However he stated that "the actual residual variance is usually 30-50% higher than the binomial variance." These studies were necessary since he did not randomize the observations. Later, (50) he recognized that random sampling erases the theoretical problems with which he was concerned and that the binomial theory is applicable when the observations are randomized. Tippett's development of snap-readings was made simultaneously with the extensive survey sampling methods referred to earlier in this chapter and hence does not reflect the findings which have emerged in that area.

The literature gives no indication that Tippett's procedure was used extensively during the years immediately following its publication. It was not until 1940 that the procedure was introduced into the literature of this country. This came as a result of an investigation by R. L. Morrow to determine the feasibility of using the technique in determining delay allowances for use in conventional time study. The time study procedure then in use consisted of carefully studying the job in order to determine the total time necessary to perform the actual work involved and the subsequent addition of a percentage allowance for delays. This procedure vitiated the accuracy of the results since there was no clear way to determine the true time for delays. The procedure advocated by F. W. Taylor with the inception of time study practice was to take continuous time studies for long periods and from these studies determine the proper allowances. In searching for a better way to determine delay and variation allowances, the method used by Tippett in the textile industry was put to test in three separate studies by graduate time study students under the direction of the Department of Administrative Engineering at New York University (39, 40). Morrow reported the findings of these studies and published the resulting method under the title "ratio-delay" (39). The method gained wide recognition in this country as a means for determining delay allowances in work measurement practices.

Morrow's research stimulated interest in the method of activity sampling and during the years immediately following his studies, several applications of the technique appeared. Abruzzi was active in making applications of the method for determining delay allowances and also published some articles pertinent to the improvement of the technique (1, 2, 3, 4). Of Abruzzi's contributions, the most significant appears to be a brief presentation in The Management Review (3) in which he advocates the use of a fraction defective control chart (p-chart) for the purpose of determining whether or not the delay percentage on a given activity is stable over extended periods of time. The concept presented by Abruzzi is in no way different from that used in quality control, but it does serve to determine the applicability of binomial theory for setting confidence limits on the estimates obtained from ratio-delay studies.

In 1950 Barnes and Correl (17) sought to "determine the reliability and validity of the ratio-delay method" by making applications to several types of industrial operations. The method used to determine validity consisted of making ratio-delay studies and time studies simultaneously on chosen operations and comparing the results. Reliability was determined by taking two ratio-delay studies simultaneously on the same operation and comparing the results. The findings of these studies showed ratio-delay analyses to be both valid and reliable. The same authors suggested a broader use of the tool by offering it for measuring productivity indexes, efficiencies, and costs. This was one of the earliest indications of the broad potentialities of the tool, the only prior indication being that of Schaeffer (45) who presented a paper in 1941 advocating its use for making cost analyses.

From the latter part of 1950 to the early part of 1952, the use of activity sampling in industry increased at a rapid rate. Numerous articles relating this widespread application appeared in publications such as Advanced Management, Agricultural Economics Research, Factory Magazine, and Management Review as well as in the proceedings of several prominent professional organizations. These articles dealt primarily with applications and procedures and seldom dealt with the theory underlying ratio-delay. The widespread acceptance of the tool as evidenced by the increased frequency of its use prompted the editors of Factory Magazine (viz, H. L. Waddell) to review the technique in an editorial in their July 1952 issue and recognize its broad applications by renaming the tool "Work Sampling." The editorial emphasized the use of Work Sampling for analyses in the general areas of product quality, machine operations, and human activity; however even broader suggestions were made by Brisley (14) in the same issue of the magazine in a compact review of the development of work sampling theory. Torgensen (51) gives Brisley credit for showing the real potential of work sampling in this article. Brisley's analysis dealt with showing the broad potential of work sampling rather than with improvements in the theory. He used the same methods for determining estimates and their variances that Tippett set forth in his early articles on snap-readings.

Publications on work sampling applications during the period 1952-1958 were twice as numerous as those in the remainder of the entire history of the tool. As in earlier years, articles continued to appear in publications which catered to the average supervisory employee. Applications of the technique which were set forth in these publications were from a wide variety of disciplines and were, in general, direct applications of Tippett's and Morrow's findings where the estimates obtained were assumed to follow the binomial law and the sampling procedure was that of simple random sampling.

That the theoretical aspects of the tool were in need of further study was recognized as early as 1953. This is reflected in Davidson's review of the tool (19) in which he depicted both the past and present state of theory and practice. His general observations were critical of the analytical procedures used in the practice of work sampling. He postulated that some of these shortcomings could be remedied through the use of more sophisticated tools of analysis, such as the analysis of variance, nonparametric methods, and binomial probability paper. As of the date of this study little has been done to improve the theoretical framework of work sampling.

Of great significance in the history of work sampling is the effort which has been put forth to justify the technique as a method for setting direct labor time standards. There are some who strongly advocate the use of the tool for this purpose (8), yet there are others who strongly disagree with this view (30). Those who do not feel that these standards should be set using work sampling feel that it should be classified as a tool for making gross measurements of complex, nonrepetitive type activities and that stop watch time study, as well as other procedures, should be classified as tools for measuring work of a narrower scope. R. M. Barnes maintains that the technique is valid for setting time standards and as a result of extensive research, he and his colleagues have presented several papers in support of his claim (6,7,8, 9,10,11,12,13,17).

The approach taken by Barnes consists of measuring "working time" and "idle time" of an activity using work sampling and employing additional information such as total time expended and total units produced to calculate the "standard time" per unit. Basically, this is the same work sampling procedure which is used in the determination of allowances, efficiency indexes, etc., but with an auxiliary calculation to determine the average time spent per unit of production. It is apparent that a time standard set in this manner does not provide the same information as does one set from time studies. In time study, the elemental breakdown of jobs permits the accumulation of data which are more refined, and which yield necessary and useful information in methods analyses, line balancing, etc. While it is possible to measure more refined segments of jobs using work sampling, as the refinement increases so does the number of readings necessary for an acceptable accuracy. Costs of such studies reach prohibitive proportions rather rapidly. In this respect, it appears that work sampling will serve the most useful purpose in work measurement if it is considered as a supplement rather than as a replacement for stop watch time study. This does not refute Barnes' major conclusion that "work sampling provides a valuable tool for measuring work," and that this simple, inexpensive method "will find wide use in many areas where manual work is performed" (8). It simply makes his statement more specific as to which of the "many areas" are more susceptible to analysis by work sampling.

In an unpublished paper (38), Moder and Gaver consider a procedure for combining attribute and variable measurements in work sampling to estimate "element life." By observing "remaining life" of short elements when work sampling observations are made and estimating element life from this data, they reduce the sample size by a factor of almost four. They feel that procedures such as theirs may ultimately be the compromise needed between work sampling and time study which will incorporate the good points of both tools.

Barnes and his colleagues have also done considerable research in the study of "performance sampling in work measurement." This effort resulted from a need to rate the performance of workers by using sampling data. They established a "Performance Sampling Project" for the explicit purpose of "providing an accurate and economical method of determining, with a preassigned reliability, the performance level of labor activities." The resulting study is discussed by Barnes (8) under the following three phases:

I. The investigation of the type of statistical distribution which is representative of performance levels of manually paced operations.

II. The development of a specific statistical technique based upon the appropriate distribution and applicable to work measurement.

III. The testing of the validity of the proposed statistical method by studies actually carried out in industrial organizations.

Phase I consisted of the collection of several observations on performance indices and the examination of the frequency distribution of the data for conformance to some known distribution. The data most closely resembled a normal distribution, however a chi-square goodness-of-fit test rejected the assumption of normality. Barnes' conclusion was that "with the rejection of the hypothesis of normality, it was decided that no generalizations could be made about the distribution of performance indices."

In view of the results of Phase I, the approach in Phase II was to use nonparametric methods. Using Tchebycheff's Inequality which states that regardless of the nature of the density function of a random variable, the probability of an observation being farther than k standard deviations from the mean is less than $1/k^2$, and applying this to the distribution of sample means of performance indices, the relationship $1/k^2 = \sigma_{\bar{x}}^2/d^2$ was obtained, where d is the difference between the population mean and the sample mean. Replacing $\sigma_{\bar{x}}$ by $\alpha/\sqrt{n}$, an equation for n, the number of observations necessary for a $1 - \alpha$ confidence level that the average performance index is within $\pm\, d$ of the true mean $(\alpha = 1/k^2)$, was found to be

$$n = \sigma^2/d^2\alpha. \tag{6}$$

This sample size must necessarily be an approximation since the standard deviation is unknown. To use this approach, Barnes states that an estimate of the standard deviation is obtained, the sample size calculated, and the sample drawn, from which a confidence limit can be set on the standard deviation. He states that the level of accuracy specified is obtained if the upper confidence limit is less than the estimate of the standard deviation chosen. This practice results in a sample size calculated from an overestimate of the standard deviation, which assures the desired accuracy. While the approach is valid, the decision maker pays for such assurance in terms of excessive sampling. For example, if $d = 2$, and $k = 2$, n is simply σ^2 and a 20% overestimate of the standard deviation results in a sample 44% too large. This is a severe penalty to pay for high levels of confidence in the estimates.

Phase III of the study was carried out by comparing performance sampling with time study. A large number of studies was conducted in eight different industrial organizations. The procedure was to have the same observer time study an operation and subsequently study it by work sampling in order to get a sampling estimate of the performance index of the operator. This approach was used to eliminate one possible source of error -- differences between raters in their concept of normal performance. The close agreement in time standards from time study and work sampling supported the hypothesis that overall standards from work sampling are reliable.

As stated earlier, the bulk of work sampling literature is technique oriented. The fundamentals of the procedure using binomial theory and simple random sampling have been published in a large number of publications serving various segments of society. However, since the tool has become widely recognized as being used more in industrial engineering than in any other single discipline, it is noteworthy that the more significant contributions to the expansion and understanding of the theory of work sampling have been made by industrial engineers. The first article pertinent to work sampling which was published in the Journal of Industrial Engineering (35) was one which explicitly stated the "standard" procedure using binomial theory. That paper reported the results of research at the University of California under the Industrial Logistics Research Project. The research was conducted for the purpose of investigating the feasibility of work sampling in general. Several other articles published in the early 1950's were also connected with University of California research projects. Sammett and Malcolm presented two papers in 1954, one dealing with applications of work sampling (34), the other with criteria for analysis and accuracy (44). In the first article they point out the growing scope of work sampling applications in such areas as equipment and operator utilization studies, work simplification analyses, the determination of element and production unit time requirements, studies of the flow pattern of materials handling, studies of jobs involving team methods of operation, and numerous others. In the second article, their objective was "to consider the relatively narrow problem of statistical tests of reliability in situations where the binomial law is applicable." This article deals with setting confidence limits and calculating sample sizes when simple random sampling is used and the binomial law is applicable. The authors also point out a need for further theoretical research for the purpose of determining which of the newer statistical procedures have application in work sampling. Specific mention is made of the analysis of variance and regression theory, but no attempt is made to evaluate any technique other than the use of binomial theory.

While the entire scope of Barnes' "Performance Sampling" as discussed above was published in the Journal of Industrial Engineering, it was not until two years after Sammett and Malcolm's suggestive remarks that studies appeared in the publication using analysis of variance concepts. Cote and Scott (18) of the Purdue Statistical Laboratory sought to compare all day time study and work sampling through the use of analysis of variance. Their intention was to test the following assumptions generally made in work measurement:

(a) That the process is stable; i.e., that the daily variability is small.

(b) That the one, two, or more machines or work stations studied are typical of all machines or work stations performing that specific job, and if a rate is set or work load capability estimated for this small sample, it should be applicable to similar machines or work stations.

(c) That the variability among the different operators or crews is negligible or can be reduced to a negligible value by rating.

As Cote and Scott point out, if work measurement is used for the purpose of estimating future capabilities or requirements, it is important that the process be stable or nearly so. They also make the statement:

> One interesting and very pertinent point is that the use of the binomial formula alone to compute the accuracy level obtained in a work sampling program ignores sources of error of larger magnitude, namely the variation among men, and the day-to-day variation of the men.

Their approach using analysis of variance techniques takes these sources of error into account. The study was conducted for three specific purposes, which were: (1) to make estimates of the average proportion of time per man per day spent on various functions of the work, (2) to estimate the magnitude of the components of error, and (3) to compare the results of the two types of sampling (work sampling versus all day time study) for differences in bias. The basic measurement made on the ith worker on the jth day was the proportion of time x_{ij} that the worker spent performing the given activity. This was considered to be the following sum,

$$x_{ij} = \mu + w_i + d_j + (wd)_{ij}$$

in which

μ = the general average over all workers over all days,

w_i = the average amount worker i deviates from μ

d_j = the average amount day j deviates from μ, and

$(wd)_{ij}$ = worker i's residual variation on day j.

Using a well known approach to analysis of variance, the authors proceeded to make observations using both all day time study and work sampling. Estimates of the components of variance σ_w^2, σ_d^2, and σ_{wd}^2 were calculated under each type of observation and compared. The general conclusion of the study was that work sampling was superior to all day time study because estimates of time spent on the activities involved could be obtained with equal error variances when time spent on work sampling was only half that spent on time study.

Numerous publications appeared during the mid-1950's on applications of work sampling but only a few dealt with theoretical problems of the tool. Moder and Halladay (37) studied the problem of a variable labor force on long cycle operations. They concluded that the effect of the correlation of workers activities on the sampling error of estimates obtained could seriously affect the reliability of the results.

Two books on work sampling were published in 1957 and both advocated the application of binomial theory and simple random sampling (8, 32), with no treatment of the gains possible through other sampling methods. One of the books (32) does recognize that improvements in work sampling are possible from this source.

Conway, in an article titled "Some Statistical Considerations in Work Sampling" (16) enumerates alternative ways of obtaining reduced sample sizes in work sampling. In effect, he suggests the use of stratified sampling procedures and also considers the case where observations are made in groups when subjects within the group are not independent. He further points out that if the activity being sampled is completely free of cyclic behavior, systematic sampling may be used for providing a more effective method of data collection. The points made by Conway are suggestive of ways to improve the theoretical aspects of work sampling, but they appear to have gone unheeded in practice due to a lack of further development.

Gambrell and his students at Arizona State University (25, 26) have done extensive research on the use of memomotion concepts in work sampling. The procedure utilizes a motion picture camera for recording observations, and results in (1) a permanent record, (2) elimination of purposeful operator behavior, (3) reduced subjectivity, and (4) reduced study costs. Although they found the procedure to have merit in some respects, they also found that it is severely hampered due to the small area covered by the camera, and none the less significant, the inability to identify the activities properly on the film. Further study by this group resulted in the development of a random actuating device which allowed random samples to be taken with the camera instead of the earlier periodic sample. Again these efforts to improve the application aspects of the technique produced no advancement in the theory. Williams (52) and Bariament (5) also discuss the application of memomotion techniques as well as the usual approach to work sampling with comments on a comparison of the two methods. There is little difference in the theory which has been applied in each case; however the administrative advantages pointed out above are sometimes significant.

No doubt the most extensive work sampling study to be found in the literature is one reported by Mindlin (36) of the Social Security

Administration. The proportions of time spent in eighteen measurable and six nonmeasurable activities of work involved in carrying out the administration's duties were sought. The area of the sample included the United States and its territories and involved 600 district offices made up of thousands of employees. Stratification was used to increase the statistical validity of the results. An "ultimate stratum" consisted of a region, a group size, a position type, and a time division, the latter being a classification of in-office, out-of-office, overtime, etc. In addition to stratification, cluster sampling was employed, with sampling at two stages. Stage I consisted of choosing m district offices at random and stage II was the selection of t_i observations on the activities of the ith office. Each office was considered as a cluster. While the complexities of calculation presented by Mindlin seem formidable, it is significant to note that this is a large scale application of work sampling which utilizes several of the advanced sampling methods of survey sampling. This study supports the thesis that sample survey methods can be applied in work sampling to improve the efficiency of sampling as well as broaden the coverage of the tool.

Rosander of the Internal Revenue Service has made significant applications of work sampling for purposes of cost estimation and control (42, 43). He employs elementary concepts of random and cluster sampling in sampling employee-minutes and cluster-minutes in order to estimate salary costs of various activities for budgetary purposes. The observations made were the costs of one minute's work and the result from the study was an overall dollar estimate of labor costs. Theoretical aspects of the estimates were not different from the usual binomial theory, although he recognized that the variances of estimates from cluster-minutes should employ appropriate cluster sampling theory. The significant deviation from the usual work sampling study is that the parent population was a cost array rather than an activity array.

Halsey of General Foods Corporation has presented a model for work sampling called the GREDS theory (29). This method places emphasis on studies in firms which cover a large geographical area and in which groups of workers observed simultaneously are not independently employed. The latter point invalidates the use of binomial theory and due to concessions often made in the theory where route observations are made, the coverage of large areas often violates the purely random pattern for observation which the binomial theory demands. The essence of Halsey's new theory is the use of ratio estimates, a method from survey sampling, to determine the proportion of time spent performing the various phases of an activity. In order for the theory to be applicable, he divides the geographical confines of a plant into unequal subdivisions which have approximately equal numbers of employees in them. The subdivisions are called GREDS (instead of grids, which indicates equal subdivisions) and observations are made on the number employed in the activity under study. The ratio estimate is p_i, the number employed in the activity divided by the number in the ith gred at the time of observation. The overall estimate of the proportion sought is then simply p, the sum of all observations on persons performing the activity divided by the sum of all observed persons. The expected value of this estimate is not quite the same as the expected value of the unbiased binomial estimate; however the bias is negligible for moderate sample sizes. The variance of the estimate must be calculated using the theory of ratio estimation. The usual procedure is then followed in setting confidence limits on the estimates.

Publications pertinent to work sampling reached a peak in the mid-1950's (in number) and have been diminishing since that time. As the foregoing review has indicated, a few studies have been made which utilize specific aspects of survey sampling, but no comprehensive study of broad applications of this theory to work sampling has appeared. It is the primary purpose of this research to improve the sampling procedures and subsequent data analyses for work sampling studies by employing methodologies from the field of survey sampling.

CHAPTER III

SIMPLE RANDOM WORK SAMPLING

The Scope of this Chapter

This chapter deals with the first specific objective set forth in Chapter I, and presents an introduction to sampling terminology and notation in terms of the simplest kind of sampling model, simple random sampling. Simple random sampling can usually be bettered for most sampling problems. The more complex sampling plans represent various deviations from simple random sampling. These deviations are introduced for a single reason -- to increase the efficiency of gathering a desired amount of information and thus minimize the expenditure of resources. Succeeding chapters will deal with the use of various sample designs in work sampling other than simple random sampling. This chapter will depict the current major approach to work sampling as an application of simple random sampling along with an explicit statement of the necessary simple random sampling theory, including its limitations. To facilitate compatability with the models developed later, a simple random work sampling approach is presented in a notation which is used throughout the study.

Some General Comments About Sampling

In any situation, the decision to gather information via the use of sampling, results from the fact that certain economies may be gained by doing so. In 1953, H. C. Grieves (31, Vol. I, Foreword) stated that hundreds of millions of dollars have been spent during the past twenty-five years in this country for statistics. Increased emphasis on statistical analyses is evident from the number of disciplines now requiring formal training in statistical methods. These facts indicate the extent to which modern business depends on statistical data. Since sampling is the basis for all statistical estimation, it plays an ever increasing role in the solution of problems in almost all phases of business, government, and science.

The manner in which data is collected in a sampling scheme determines its usefulness. The errors of estimates calculated from samples are functions of the sample design, and since the former are of utmost importance in any statistical analysis, the sample design becomes the key to sampling success. In order to design efficient sampling procedures, one must consider the efficient use of all available resources, including information about the population to be sampled. The optimum design is the one which combines the physical facilities, personnel, and available statistical information about the universe being studied in a manner which minimizes total expenditures of resources while gaining the desired information. It is almost always possible to use several alternative sample designs in collecting data for a given problem, but one must be able to make a rational choice among these if he is to produce an efficient plan.

It was noted in the literature search that the primary difference between early sampling methods and modern ones is the use of probability theory. Without the latter there is no method which will allocate resources to a sampling scheme in an optimum fashion. With probability theory, the estimation of the variance of sampling errors is possible if the sampling scheme which was employed is known. With this information the precision of results from different schemes may be compared and the design chosen which will yield the desired precision at minimum cost.

The Selection of a Sampling Unit

Where sampling is appropriate there exists an aggregate of "units" about which one wishes to have certain information, but which one does not wish to inspect individually. It is assumed that there are N such units and that as an aggregate they shall be called a universe (or population). The size of the population may be finite or infinite. To gain estimates of population characteristics by using a sample, one selects a portion of the units in the universe for study. The portion selected is called a sample and will contain n of the total N units. The "plan" adopted for selecting the n units is known as the sampling plan.

As pointed out earlier, the units chosen in the sample must be selected randomly from those available for selection if probability theory is to be applied in analyzing the data. Only randomly selected samples, as opposed to systematic or haphazard sampling, will be considered in this study. If the population size, N, is finite, then a random sample of n units from the population is one which is chosen such that each of the $\binom{N}{n}$ combinations of n units have the same chance of being selected. If, however, the number of units in the population is infinite, the sample of n observations will be a random sample if the n observations are independent and if each unit in the population has the same chance of being included in the sample at each selection.

The aggregate of units in a work sampling situation (the population) is the total time period of the study. The need arises for specifying what the individual "unit" in the population will be for the purposes of sampling. To apply probability sampling theory for making estimates of population parameters, and for calculating the variances of such estimates, it is necessary that the unit ultimately sampled be the basic unit in the population on which the estimates are to be made. Historically, work sampling practitioners have selected minutes as the basic sampling unit. The theory applied for making estimates from the data has been that which assumes an infinite population. Actually, if minutes are chosen as the sampling unit, then the population sampled consists of a finite number of minutes and a closer investigation is necessary to determine whether or not the infinite theory is applicable.

The interval of one minute has obviously been chosen for practical considerations. It is expedient to choose an interval such that randomization of the observations can be conveniently carried out. One minute appears to be about the smallest time unit for which this will be practical. The use of infinite population sampling theory is justified in the following way. The interval chosen (one minute) is considered to be a period in which one will observe the activity at some random instant, the randomization of instants being left to natural phenomena. The number of instants in the entire population obviously approaches infinity. This practice precludes having to randomize the observations on "instants," which would be administratively uneconomical. It provides a random sample of instantaneous observations, and allows the use of infinite population concepts.

Using the scheme of first choosing a random minute and then choosing a random instant within the chosen minute yields the same results as initially choosing a random instant, if the scheme is properly administered. The most important requirement is that each minute in the entire population be subjected to sampling each time a random minute is chosen, i.e., the minutes are sampled "with replacement." Otherwise, the final sample of n instants will not be a random sample from the population of N instants, a necessary requirement in the analytical phases of the analysis. Occasionally this procedure will result in some minutes being chosen more than once in a given study. This will require that more than one random instant be observed within such minutes. Where a single observation is made within each chosen minute the randomization within the chosen minute is left to natural phenomena, such as the variability in the exact time of the observer's arrival at the activity, etc. In case more than one observation is made within a chosen minute, these may be randomized in the same manner in which the chosen minutes were randomized. For example, two observations within the same minute may be randomized by choosing two random numbers between 1 and 60 (representing seconds) and the difference between these numbers noted, say d. The first observation could then be made in the usual fashion and the second one made d seconds later. Technically, this is equivalent to adopting one second of time as our sampling unit. However, in all cases of practical interest, this makes the finite population correction factor essentially equal to one and can be assumed as such.

The frequency with which the same minute is sampled more than once depends on the "density" of sampling, i.e., the ratio of n to the number of minutes in the population, say U. In work sampling studies where n, the sample size, is only a small fraction of U (the usual case), the total number of minutes in the population under study in which two or more observations will occur will not be great. For example, if a study is made which required that $n = 500$, and U is 10,000, the expected number of minutes within which two or more observations will be required will be approximately twelve (see Feller, 22, page 94).

If sampling units longer than one minute were chosen, the natural causes of randomization within the chosen interval would not be as operative, and the assumption of a random sample of instants would be considerably weaker. Therefore, it appears acceptable to choose the sampling unit of "one minute" as it provides the necessary practical features as well as the necessary theoretical setting for analyzing the data.

Throughout this study, we shall assume that the sampling unit of one minute will be used for administrative expediency. Within a chosen minute, we shall assume that the activity is observed instantaneously, and that the number of instantaneous observations possible within the period of a study is infinite. These assumptions are not absolutely true since an "instant" of time must have finite dimensions. However, the aggregate number of instants will be so large for a given study that finite and infinite sampling theory will be essentially the same.

Simple Random Sampling

In order to illustrate various aspects of the sampling plans presented throughout the remainder of this study, a typical work sampling universe is illustrated in Appendix A. This appendix represents an activity array of a hypothetical work situation in which there are four individuals engaged in an activity consisting of three separate elements (or states). The total span of the illustrated activity is two weeks, or ten working days. In this case, the population has been conveniently represented in man-minutes. The elementary sampling unit is an instant as described in the preceding section. The universe contains an aggregate of $U = 19,200$ man-minutes and an infinite number of instants. A simple random sample of minutes can be selected a number of ways; however, the most convenient method appears to be by the use of a random number table. The procedure for accomplishing this will be detailed in a later section.

Estimators from a Simple Random Sample

The notation adopted for this study is as follows. Universe properties will be denoted by capital letters and sample properties will be denoted by small letters. The N elementary units in the universe will be assumed to be numbered from $i = 1$ to N. In most, but not all cases, the sampling procedures presented in this study will be developed under the supposition that N is extremely large. Finite population concepts will be used in defining the estimators, their variances, etc., in which there are N units in the population and n of these have been chosen in the sample. When the proper statements have been made in terms of N, a finite population, the limit as N approaches infinity will be taken as the proper expressions to use in work sampling analyses dealing with infinite populations. Characteristics of units in the universe which are to be measured will be denoted by letters such as X, Y, and Z. Specifically, for the ith unit in the universe, these would be X_i, Y_i, and Z_i. Likewise, the measured characteristics for the jth unit in the chosen sample will be x_j, y_j, and z_j, $j = 1, 2, \ldots\ldots, n$.

In general, the characteristics of the units to be studied may be any which are capable of being defined. The most common parameters of universes which one wishes to study are averages per unit, totals for the entire universe, and proportions of the universe which possess certain characteristics. The latter is the measure of most interest in work sampling. The total of a specified characteristic, say X, over all members in the universe will be

$$X = \sum_{i=1}^{N} X_i = X_1 + X_2 + \ldots\ldots\ldots + X_N. \tag{7}$$

The mean of this characteristic for the entire universe will be

$$\bar{X} = \sum_{i=1}^{N} X_i/N = X/N . \tag{8}$$

In the same manner, the total for a sample of n items will be

$$x = \sum_{j=1}^{n} x_j = x_1 + x_2 + \ldots\ldots + x_n, \tag{9}$$

and the sample average will be

$$\bar{x} = \sum_{j=1}^{n} x_j/n = x/n . \tag{10}$$

The sample values are used for estimating the universe parameters and are called estimators. A specific realization of an estimator will be called an estimate.

For proportions, which are of primary importance in work sampling, the objective is to estimate the proportion of the N basic units in the universe which possess a given characteristic. If $X_i = 1$ when the unit possesses the characteristic and $X_i = 0$ otherwise, then the following notation will suffice,

$$X = \sum_{i=1}^{N} X_i = \text{Total number of units in the universe possessing the Xth characteristic,} \tag{11}$$

$$P = \sum_{i=1}^{N} X_i/N = X/N = \text{Proportion of units in the universe possessing the Xth characteristic.} \tag{12}$$

For the sample,

$$x = \sum_{j=1}^{n} x_j = \text{Total number of units in the sample of } n \text{ units possessing the Xth characteristic,} \tag{13}$$

$$p = \sum_{j=1}^{n} x_j/n = x/n = \text{Proportion of units in the sample possessing the Xth characteristic.} \tag{14}$$

Therefore, an observed value of p is used as an estimate of P. Note that $Q = 1 - P$ is the proportion of the universe not possessing the characteristic and $q = 1 - p$ is the proportion of the sample not possessing the characteristic. These terms are introduced for later use.

Properties of the Estimators

The estimators $\bar{x}$ and p have certain properties which are well known in mathematical statistics. They are consistent,* meaning that as $n \to \infty$, the distribution of the estimate becomes more concentrated about the value being estimated; they are unbiased,* which means that $E(\bar{x}) = \bar{X}$ and $E(p) = P$, where E( ) is the expected value of the variate in parentheses. These are desirable properties of the estimators used in work sampling but will not play a major role in this study since most of the estimators used are unbiased and consistent. In fact, the estimators $\bar{x}$ and p are sufficient statistics* which means that they summarize all the information in the sample about the parameters of interest so that no other statistic will provide additional information. Unless otherwise stated, the estimators used will possess these properties.

The measure of precision commonly used for estimators in statistical analyses is their standard deviations. By definition, the standard deviation of an aggregate of measurements (a universe) is

$$\sigma_x = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N}} \tag{15}$$

or

$$S_x = \sqrt{\frac{N}{N-1}\,\sigma_x^2} = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N - 1}} \tag{16}$$

*See Fraser (24, page 215ff).

The latter form simplifies some of the formulas developed later. The sample standard deviation is denoted by

$$s_x = \sqrt{\frac{\sum_{j=1}^{N} (x_j - \bar{x})^2}{n-1}} , \tag{17}$$

and is a useful estimator of the universe standard deviation. Using this notation, the variances of $\bar{x}$ and p are found to be*

$$\sigma_{\bar{x}}^2 = \left(\frac{N-n}{N}\right) \frac{s_x^2}{n} \tag{18}$$

and

$$\sigma_p^2 = \left(\frac{N-n}{N}\right) \left(\frac{N}{N-1}\right) \frac{PQ}{n} . \tag{19}$$

Since p is a special case of $\bar{x}$, these two formulas are equivalent. It should be noted that for extremely large populations, i.e., as $N \to \infty$, the product of the terms $(N-n)/N$ and $N/(N-1)$ approaches unity, since n is always finite.

Another measure of precision which should be noted due to its frequent use in sampling literature is the coefficient of variation. Where the foregoing measures of precision, $\sigma_{\bar{x}}^2$ and σ_p^2, are absolute measures of the precision of $\bar{x}$ and p, respectively, the coefficients of variation, defined as

---

*See Hansen, Hurwitz, and Madow (31, Vol. II, pp. 92-96).

$$V_{\bar{x}} = \frac{\sigma_{\bar{x}}}{\bar{x}} \quad \text{and} \quad V_p = \frac{\sigma_p}{P} \ , \tag{20}$$

are relative measures of precision. The square of the coefficient of variation is called the rel-variance.

Designing a Simple Random Sample for a Stated Level of Precision

The primary uses of the foregoing measures of precision are concerned with stating the sample sizes necessary for obtaining estimates which are "fairly certain" of being within a stated range of the parameter being estimated. For large samples from large populations, where the sample is not a significant portion of the population, the central limit theorem allows the assumption that $\bar{x}$ and p are approximately normally distributed.* Thus if one wants the relative difference between the sample estimate, p, and the true value, P, to be no greater than R with a stated confidence,** i.e., that $P(\left|\frac{p-P}{P}\right| \leq R) = 1 - \alpha$, one may set $k_{\alpha/2} V_p = R$, in which $k_{\alpha/2}$ is a standard normal deviate corresponding to the chosen confidence of $1 - \alpha$. Solving for n, it follows that

$$n = \frac{K^2_{\alpha/2} Q}{R^2 P} \ , \tag{21}$$

which is the required sample size for the estimate to be within the

---

*See Parzen (41, p. 371ff). Also see Cochran (15, p. 41) for limits of this assumption when the variate is p. The assumption is seldom a problem in work sampling unless the value of P being estimated is near zero or one.

**The choice of the value of R in a given case will reflect the decision makers judgment as to how large an error can be tolerated in the estimate. This is actually an economic consideration and is investigated more fully in the section dealing with costs which appears later in this chapter.

stated limits of error, R, at the chosen level of confidence $1-\alpha$. Graphs of this function for three common levels of confidence, the total range of P, and selected values of R, appear as Figures 1, 2, and 3. When the estimator is $\bar{x}$ rather than p, the calculations for n are similar to those above.

Adequacy of the Sample Variance

An unfortunate aspect of the above measures of precision, $\sigma_{\bar{x}}$, σ_p, $V_{\bar{x}}$, and V_p, and the succeeding formula for sample size, is that they involve the parameters $\bar{X}$ and P which are the values being estimated. It thus becomes necessary to estimate the variance of the estimator from the sample. The estimate of the population variance presented earlier is used for this purpose (see equation 17), and the resulting unbiased estimates of $S^2_{\bar{x}}$ and S^2_p are*

$$s^2_{\bar{x}} = \left(\frac{N-n}{N}\right)\left(\frac{s^2}{n}\right) \tag{22}$$

and

$$s^2_p = \left(\frac{N-n}{N}\right)\left(\frac{N}{N-1}\right)\left(\frac{pq}{n-1}\right) . \tag{23}$$

For infinite populations these reduce to s^2/n and $pq/(n-1)$, respectively.

Because the sample variance is a random variable, the values of $s^2_{\bar{x}}$ and s^2_p are only approximations of $S^2_{\bar{x}}$ and S^2_p. While the effect of these approximations is quite serious when estimating means, it is not a serious problem when estimating proportions (31, Vol. I, p. 133). Due

*See Hansen, Hurwitz, and Madow (31, Vol. II, pp. 98-99).

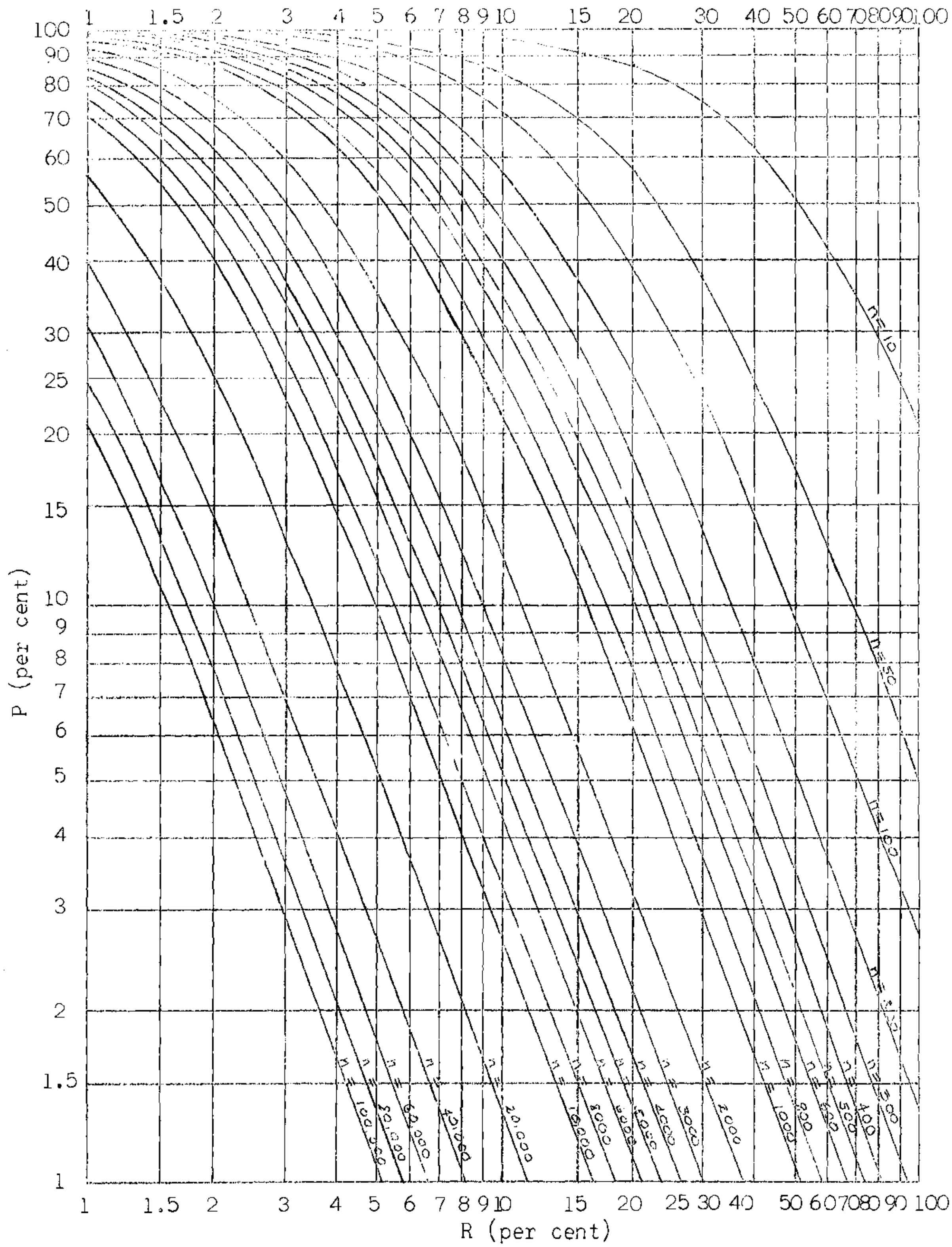


Figure 1. Curves for Determining the Sample Size, n, Necessary for 90 Per Cent Confidence that p Will Be Within 100R Per Cent of P. (For Using Average Error, $\bar{E}$, $R = 2.062\bar{E}/P$. For a Given R and P, $\bar{E} = 0.485RP$.)

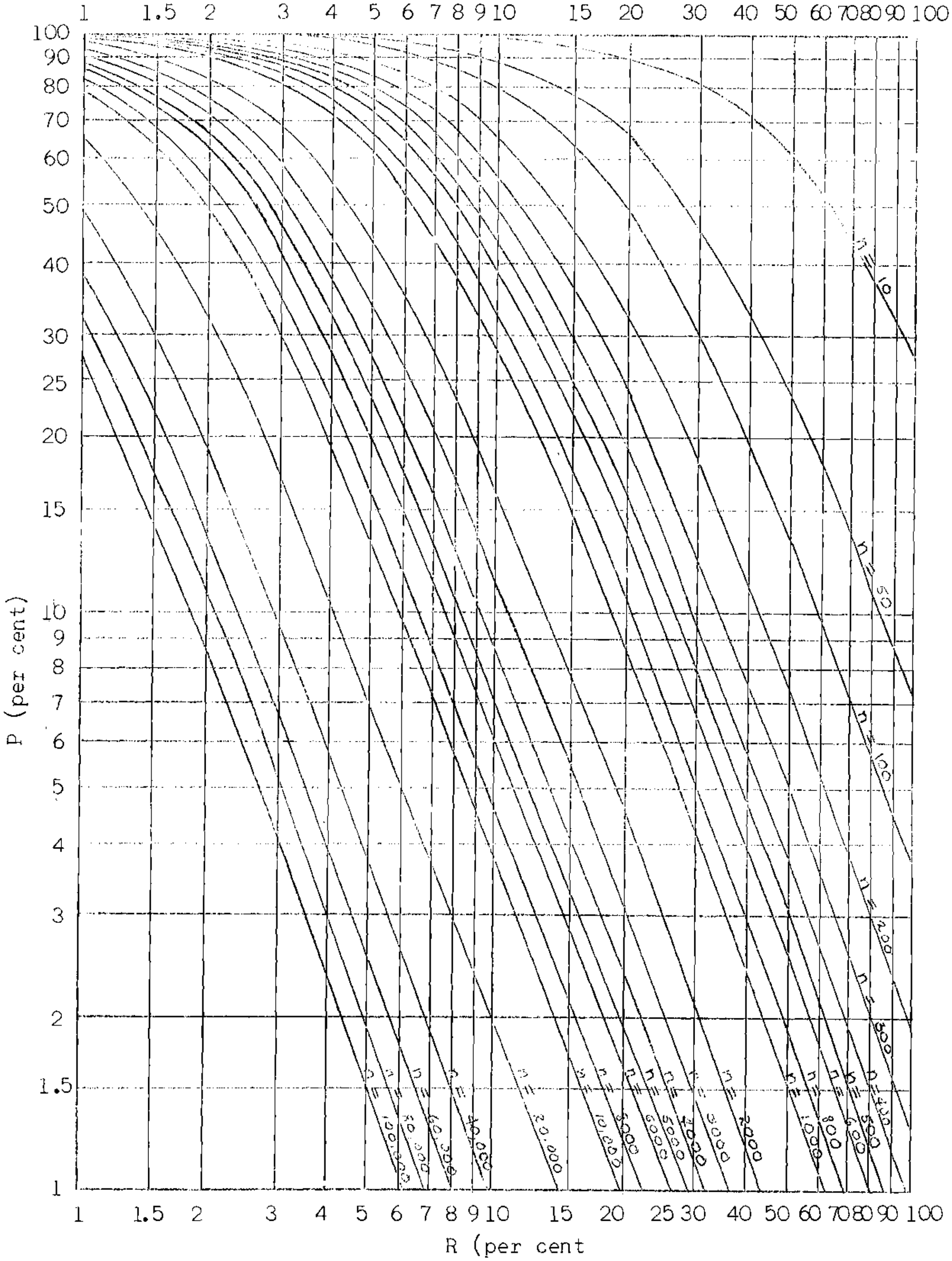


Figure 2. Curves for Determining the Sample Size, n, Necessary for 95 Per Cent Confidence that p Will Be Within 100R Per Cent of P. (For Using Average Error, $\overline{E}$, $R = 2.456\overline{E}/P$. For a Given R and P, $\overline{E} = 0.407RP$.)

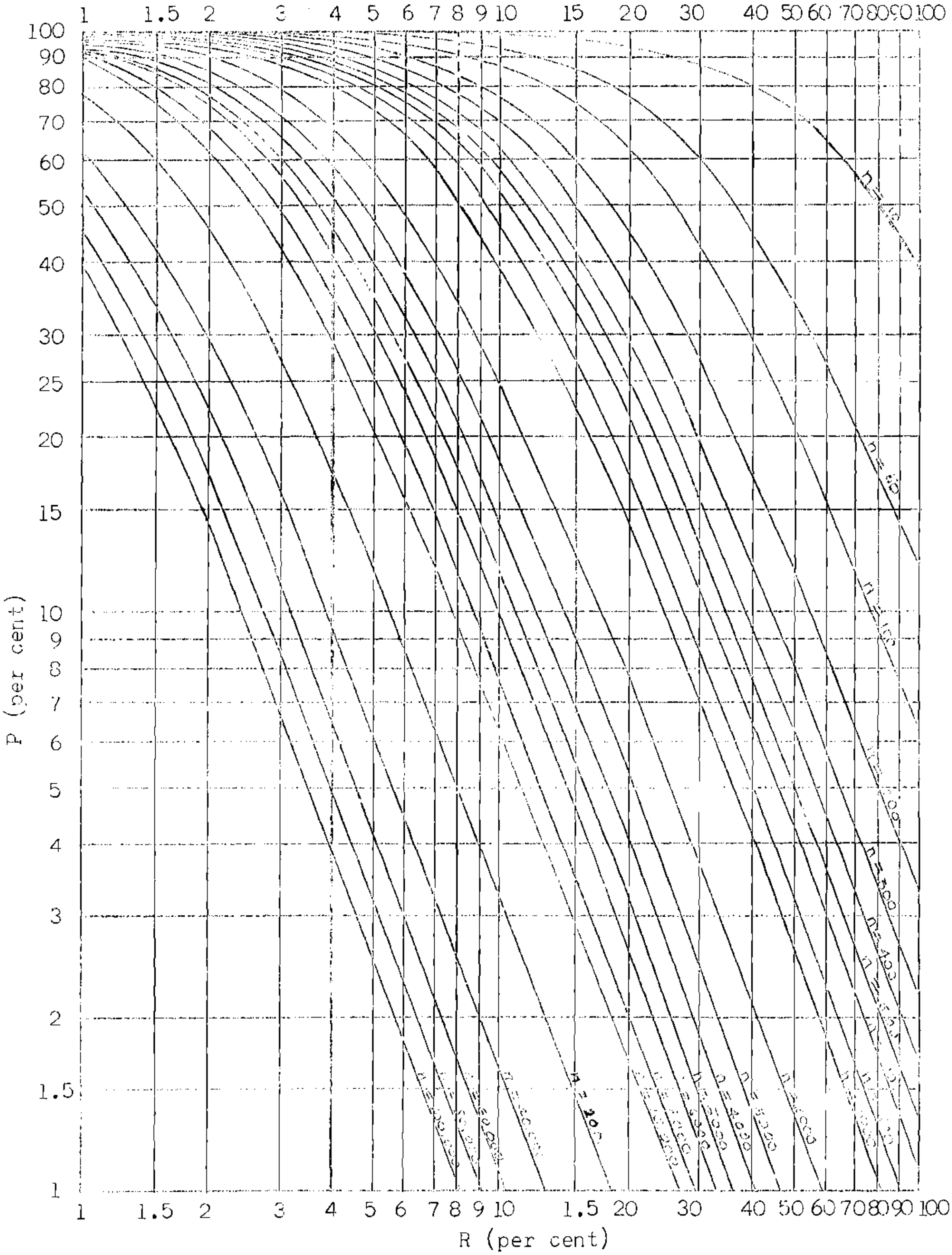


Figure 3. Curves for Determining the Sample Size, n, Necessary for 99 Per Cent Confidence that p Will Be Within 100R Per Cent of P. (For Using Average Error, $\overline{E}$, $R = 3.233\overline{E}/P$. For a Given R and P, $\overline{E} = 0.309RP$.)

to the fact that this study is concerned with improved methods of work sampling, and estimates are typically proportions, only the case of proportions will be considered here.

The problem encountered is that the sample estimate of the variance, being a random variable, will vary from sample to sample with a coefficient of variation V_s. It would be desirable to estimate S with as small a coefficient of variation, V_s, as is practical. As for most estimators, V_s can be made smaller by increasing the sample size. If one can show that the sample taken in a given work sampling study is large enough that V_s is sufficiently small as to introduce only a negligible error into the analysis, then estimating the variance from the sample would be permissible.

It has been shown that the coefficient of variation for the standard deviation of p is*

$$V_{s_p} \doteq \sqrt{\frac{1}{4n}\left(\frac{1}{PQ} - \frac{4n - 6}{n - 1}\right)} \ . \tag{24}$$

A plot of V_{s_p} for various values of P and n will show the sample size needed for making V_{s_p} a desired magnitude for all possible values of P. The plot is shown in Figure 4.

From this figure one can see that for a specified value of V_{s_p}, the necessary sample size becomes larger as the value of P deviates farther and farther from 0.5. Survey sampling practitioners have found that values of $V_{s_p} \leq 0.1$ introduce only small errors in confidence interval estimates of the parameters when s_p is used as an estimate of

*See Hansen, Hurwitz, and Madow (31, Vol. II, p. 105).

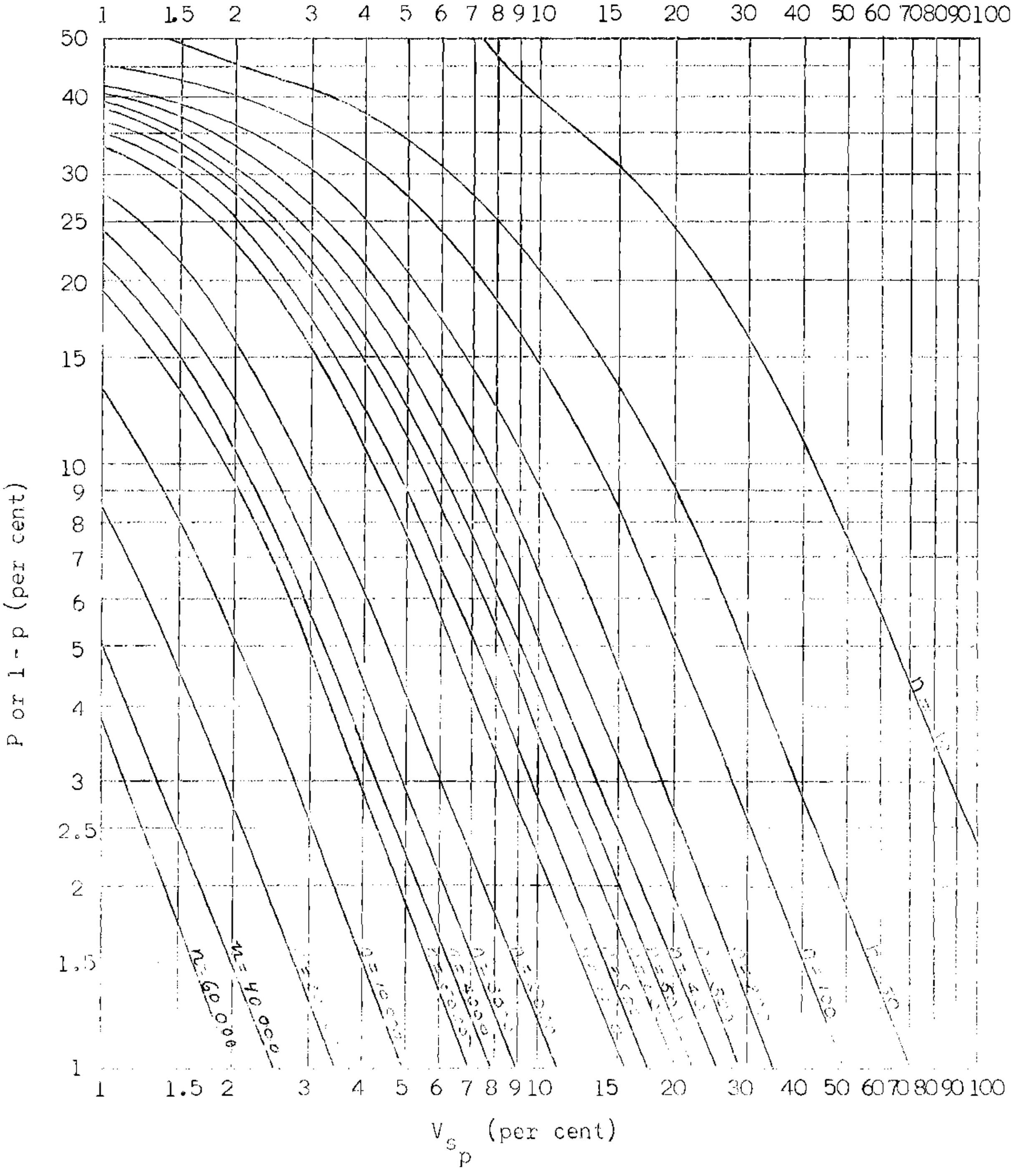


Figure 4. Variation of the Sample Variance of p for Selected Values of n

σ_p (31, Vol. II, pp. 105-106). If this value is adopted as being satisfactory for work sampling purposes,* one may readily determine whether or not the value of n calculated in the design of the sample is sufficient to insure that V_{s_p} is of an acceptable magnitude. Figure 4 reveals that if $0.1 \leq P \leq 0.9$ and $n \geq \approx 180$, then V_{s_p} will be no greater than 0.1. Since the values of P and n in work sampling studies are generally well within these limits, it becomes clear that V_{s_p} will, in most cases, be considerably less than 0.1, and s_p will be a reliable estimate of σ_p.

The estimator p is approximately binomially distributed when the sample size, n, is small relative to the universe size, N. Also, as already stated, when N and n are both very large, and n is small relative to N, the distribution of p approaches a normal distribution. Since these conditions hold in work sampling situations (the populations of instants are assumed to be infinitely large), the normal approximation of p with mean p and variance $pq/(n-1)$ is a reliable description of the distribution of the estimator p. The denominator in the variance expression, $n-1$, is frequently replaced by n for simplicity. If more accurate results were desired, one could revert to a table of hypergeometric or binomial distributions, but with considerably more work. Such tables are available when they are really needed, i.e., for rela-

---

*The value of V_{s_p} desired for a given situation is of course arbitrary, however, the real concern is that the variance of the estimator be "reasonably" correct. The rule of $V_{s_p} \leq 0.1$ is analogous to the common rule of requiring $n \geq 30$ to allow the use of the normal approximation to the t distribution in setting statistical confidence limits. The magnitude of the error in such confidence limits is negligible when n is large. The same is true of confidence intervals in work sampling which are set using a sample estimate of the variance from a large sample.

tively small values of n and N.

Simple Random Work Sampling

The foregoing discussion of simple random sampling leads to the theory which is usually assumed in most work sampling studies. The objective of work sampling studies, as pointed out earlier, is to estimate the proportion of time which is spent performing well defined elements of the activity, e.g., idle time, set-up time, working time, etc. The current procedure for conducting such studies is outlined briefly in Chapter I (pages 3-5) where the total period of the study is considered as a universe of U minutes, and observations of the state of the activity are made at randomly selected instants. If there are three states of activity under study, as in Appendix A, then one lets $x_j = 1$ if the activity is in state one when the jth observation is made, $y_j = 1$ if in state two, and $z_j = 1$ if in state three. Otherwise, x_j, y_j, and z_j are zero. To determine the number of observations necessary for a stated precision, estimates of P_1, P_2, and P_3 (proportions of time in each state) are made and equation (21) is solved for n, where P and Q are replaced by the estimated values. These estimates may be determined from prior experience with similar activities or from pilot studies. The use of data obtained in previous studies often provides adequate advance estimates of the parameters and hence eliminates the need for a preliminary sample. This source of data obviously becomes more valuable with the age of a work sampling program.

The values of n calculated for each element will of course be different and a decision must be made as to the single value which will be used. If the largest value of n is chosen, then each estimate from the sample will be within the stated limits of accuracy. This practice will result in levels of accuracy which are greater than that specified for all the estimators other than the one for which n was chosen. The accuracy for this estimator will be that which was specified. Accuracy is usually specified for the most important elemental estimates and the accuracies for the others allowed to fall where they may. The decision here depends upon the personal desires of those who will use the data. A check of Figure 4 using the values of P and n arrived at in this decision will indicate the sufficiency of the sample size for estimating the variances of the estimators. If n is too small to give an estimate of the standard deviation with $V_{s_p} \leq 0.1$, either the sample size must be increased to the minimum value for this to hold, or else the confidence interval estimates made using the estimate of variance must be regarded as being rough approximations to the correct confidence intervals, which could be constructed with considerably more computation.

Confidence Interval Estimates

In terms of the previous notation, the estimates of P_1, P_2, and P_3 in Appendix A, denoted by p_1, p_2, and p_3 respectively, and their estimated variances are as follows:

$$p_1 = \sum_{j=1}^{n} x_j/n \qquad s^2_{p_1} \doteq p_1 q_1/n \tag{25}$$

$$p_2 = \sum_{j=1}^{n} y_j/n \qquad s^2_{p_2} \doteq p_2 q_2/n$$

$$p_3 = \sum_{j=1}^{n} z_j/n \qquad s^2_{p_3} \doteq p_3 q_3/n \ .$$

Using the assumption that the estimators are normally distributed, confidence statements may be made about the values of the true P_i's by stating that

$$P(-ks_{p_i} \leq p_i - P_i \leq ks_{p_i}) \doteq \int_{-k}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} \, dr \ . \tag{26}$$

The confidence interval estimate is valid when used to estimate the individual proportions one at a time. However, due to the fact that the sample estimates are not independent, a joint confidence statement is not equal to the product of the probabilities associated with the single statements, i.e.,

$$P(p_1 - k_{\alpha/2}\, s_{p_1} \leq P_1 \leq p_1 + k_{\alpha/2} s_{p_1};\ p_2 - k_{\alpha/2}\, s_{p_2} \leq P_2 \leq p_2 + k_{\alpha/2} s_{p_2}; \ldots$$
$$\ldots\ldots\ldots\ldots;\ p_i - k_{\alpha/2}\, s_{p_i} \leq P_i \leq p_i + k_{\alpha/2}\, s_{p_i}) \neq (1 - \alpha)^i.$$

In addition, the lack of independence between the sample mean, p_j, and the sample variance, $s^2_{p_j}$, further prohibits the evaluation of the joint statement with simple probabilities.

The results of 200 simulated realizations of a three element study are shown in Table 1. The results which would have been expected had all elements been independent are shown in parentheses for comparison. The deviation from the independent case is about as expected. For example, the tendency was for fewer cases of one interval being in error and more cases of two intervals being in error, the latter being due to the dependent relationship between the estimators; i.e., if one interval tended to be in error, this tended to cause one of the other intervals to also be in error. The simulation is explained more fully in the following section.

Table 1. Simulated Results for Confidence Statements when Elements are Dependent. (All values are in per cent.)

Number of individual intervals failing to include the mean	Confidence Level 0.90	0.95	0.99
0	(72.90) 79.50	(85.74) 88.00	(97.03) 98.00
1	(24.30) 10.50	(13.54) 7.50	(2.94) 1.50
2	(2.70) 10.00	(0.71) 4.50	(0.03) 0.50
3	(0.10) 0.00	(0.01) 0.00	(0.00) 0.00
Overall	(90.00) 89.80	(95.00) 94.50	(99.00) 99.20

An Illustration of Simple Random Work Sampling

As a means of illustrating the approach to simple random work sampling as well as for providing a basis for comparing the simple designs presented in succeeding chapters, a simple random sampling plan will be designed for the activity illustrated in Appendix A. The objective of such a sampling plan is to provide data for estimating the proportion of total time spent in each of the three states of the activity. We shall work with known values of P_i throughout this study since the objective of the study is to show the efficiencies of other methods of sampling as compared with simple random sampling. The values of the P_i's and the sample sizes necessary for an accuracy of $R = 0.10$ with confidence of 0.95 are as follows:

$$\begin{aligned} P_1 &= 0.1650 \qquad & n_1 &= 1944^* \\ P_2 &= 0.2570 \qquad & n_2 &= 1111^* \\ P_3 &= 0.5780 \qquad & n_3 &= 280^* . \end{aligned} \tag{27}$$

A check of these values of n and P in Figure 4 shows each of them to be well within the limits for estimating σ_p^2 by s_p^2 from the sample. The values of P in Figure 4 represent actual values of the population parameters being estimated. These of course must be approximated for making the sample size calculations in equation (21) and if Figure 4 is used at this stage, the approximations are assumed to be actual. However, if the observed value of p is greater than $0.1 + 3\sqrt{\frac{(.1)(.9)}{n}}$ and less than $0.9 - 3\sqrt{\frac{(.1)(.9)}{n}}$, one can say that $0.1 \leq P \leq 0.9$ with almost certainty. Hence, one may take the sample of n observations and if the estimate, p, is between these limits, be quite confident that $V_{s_p} \leq 0.1$.

A compromise among the values of n calculated above must be reached before proceeding with the study. We shall use the maximum n and merely state the higher level of precision of the other two estimators.

---

*Calculated by equation (21) or read from Figure 2.

Instead of $R = 0.1$ for elements two and three, these will be $R_2 = 0.076$ and $R_3 = 0.038$. These values are obtained by setting $n = 1944$ in equation (21) and solving for R.

The selection and analysis of this sample ($n = 1944$) by simple random methods is accomplished by the following procedure (Reference Appendix A):

1. Each man-minute is specified as belonging to only one day (10 possible), one hour within a day (8 possible), and one man (4 possible).

2. A table of random numbers is used to choose a specific man-minute by letting the first digit observed represent the day chosen (1,2,.......,9,0), the second digit observed represent the hour within the day which is chosen (1,2,........,8, ignore 9 and 0), and the third digit represent the man to be observed (1,2,3,4, ignore all others). For example, the choice of 87341 from the random number table indicates that on day eight, at hour seven, observe man three at some random instant within minute 41. Other schemes, equally effective, could be devised for selecting the random minutes.

3. Proceed to choose man-minutes in this manner until the total number of observations has been specified. This step of the study may well be relegated to a computer. In case some of the chosen minutes occur in the sample more than once, randomize the observations within a given minute by designating additional random numbers to indicate the exact "instant" within the interval at which each observation will be made. This practice is necessary in order to make the simple random theory applicable.

4. Make the observations over the ten day period at the designated

times (the observations will be at random instants within each chosen minute).

5. Calculate the estimates of P_1, P_2, and P_3, and the variance of each as indicated in the preceding section.

6. Set confidence interval estimates for the P_i's as also discussed in the preceding section.

In order to test the accuracy of the simple random procedure in work sampling, two hundred random samples of size 1944 were drawn from the population in Appendix A and the values of p_1, p_2, and p_3 were calculated for each sample. This extensive simulation was carried out on a computer. The frequency distributions obtained for each of the estimators are shown in Figure 5. The actual values of the parameters and their variances are listed by each distribution. Chi-square goodness-of-fit tests support the assumption of normality for all of the estimators, and the sample values indicate that the estimate of variance is very accurate.

Sampling plans other than simple random sampling will be presented in succeeding chapters for the purpose of estimating the P_i's with greater efficiency. It is desired that for a stated cost, the precision of the estimators be increased, or conversely, for a stated precision, the total costs of the study be reduced. Elements of costs in simple random work sampling are considered in the next section.

Cost Factors in Simple Random Work Sampling

Either because the total resources available for a given study are fixed, or because it is desired to know the general magnitude of the total costs of a study, it is necessary to consider the nature of the

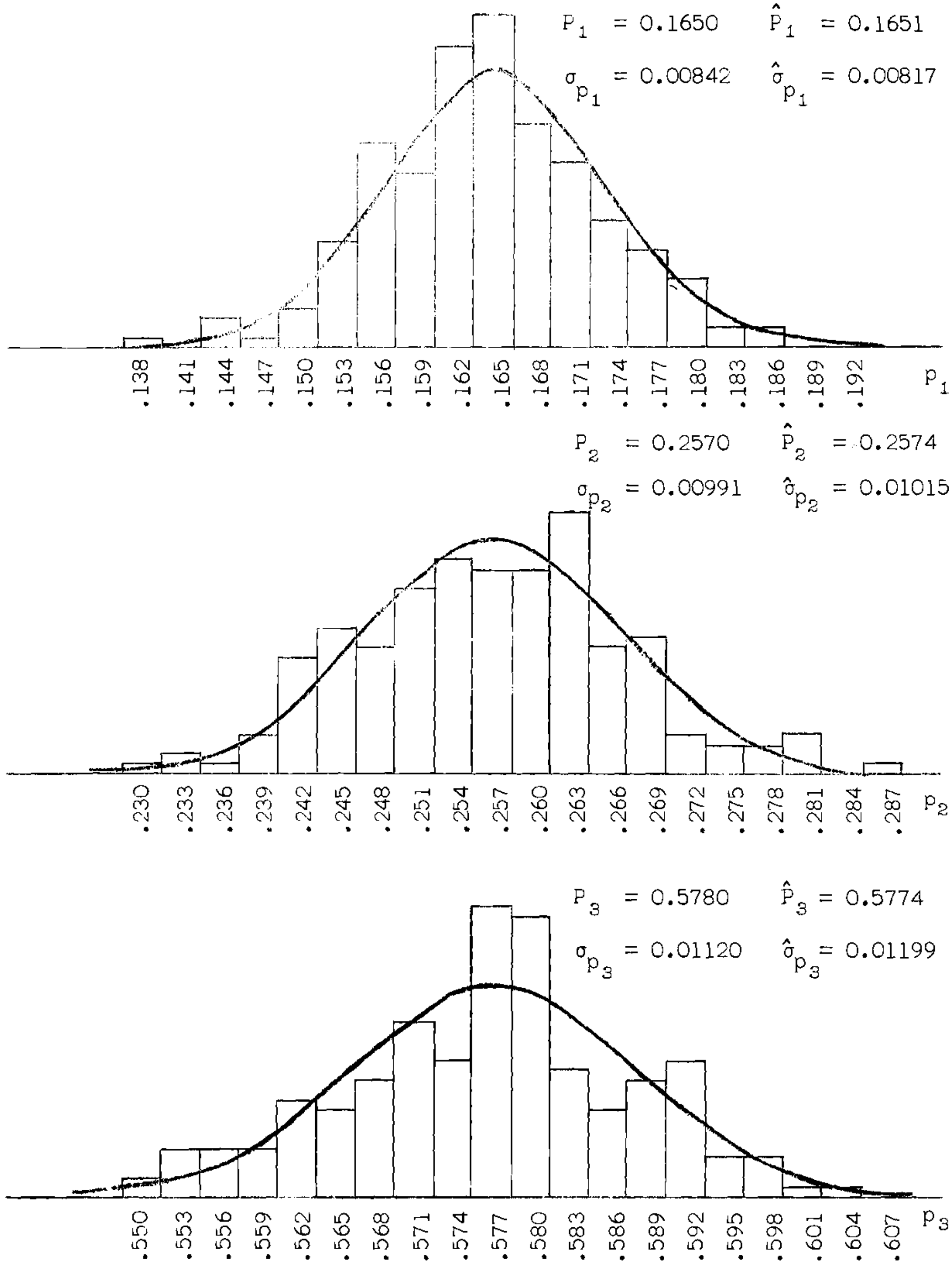


Figure 5. Frequency Distributions of p_1, p_2, and p_3 for 200 Studies Compared with Normal

costs in work sampling. For the simple random sampling approach set forth in this chapter, and in general for the procedures which follow, the following categories of costs are pertinent.

1. Fixed costs -- the costs associated with providing central administrative and technical work on the study, including the costs of space and equipment. The costs may be expected to remain about the same even though there are marked variations in the sampling design; hence, it shall be assumed that they are independent of sample size.

2. Variable costs -- those costs which evolve from assembling a sample of items from a specified universe. These costs are a function of the sample size.

3. Losses due to error -- those costs which occur as a result of the estimate of a population parameter being in error.

Designing a Simple Random Sample to Minimize Costs or to Achieve a Stated Total Cost

All costs of a work sampling study are considered to fall into one of the above three categories, hence the total cost of such a study may be expressed as

$$C = C_1 + nC_s + C_e \tag{28}$$

where C_1 is the relevant fixed costs of the study, assumed to be independent of n, nC_s is the cost of sampling (each item in the sample is considered to contribute the same amount to sampling costs), and C_e is the loss due to an error $(e = p_i - P_i)$ in the estimate.

Eliminating the constant costs C_1, the total cost for purposes of designing the sample is

$$C' = nC_s + C_e \, . \tag{29}$$

The cost C_s of a single observation in the sample can usually be determined for a given study. This cost is arrived at by considering mainly the cost of the observer's time, however it should include any other factors which contribute to the cost of each observation, particularly if the observation disrupts the process in any way. The cost C_e is a function of the error in a given estimate, say $l(e)$. The probability distribution of the error of estimate is a function of n, the sample size, and can be denoted by $g(e;n)$; it is usually assumed to be the normal distribution. The expected loss for a given sample size is simply

$$E\left[l(e)\right] = \int_e g(e;n)\ l(e)\ de = C_e \, . \tag{30}$$

Since the error, e, is the difference $p_i - P_i$, for unbiased estimators it has expected value of zero. The variance of e is PQ/n since p_i has that variance.* Therefore $g(e;n)$ can be specified. The function $l(e)$ is not only difficult to specify in a given situation, but it will also differ from study to study. It is logical however, that the loss is an increasing function of error. In work sampling, a small error is likely to be insignificant, but a large error is likely to be highly significant, especially in the case of labor standards. Hence, a loss function which increases more rapidly as e becomes larger would seem appropriate. One function which would describe such losses is ce^2, where c is a constant. With this loss function, the cost will vary from

*The denominator is actually $n-1$, but is increased to n to facilitate computation.

zero when e is zero to c when the error is one, its theoretical maximum. The actual error experienced would be expected to be near the lower limit in realistic cases. The problem of determining the constant, c, in a given case could be handled by consultation with the decision maker who will use the work sampling data and by subsequently listing values of error, e, and estimates of the costs which these errors would introduce. The loss function could then be fitted to these data in all cases, regardless of whether the form is ce^2 or some other function. An alternative way of specifying the function applicable would be to present a series of curves with different parameters and let the decision maker choose the appropriate one.

In order to facilitate the presentation of cost considerations in work sampling, the function ce^2 will be used as the "loss due to error" cost function throughout this study in a general way. If some other function were more appropriate, it could be inserted in place of the one assumed, and although the analysis may be more complex, the conceptual approach would not change.

With the loss function $l(e) = ce^2$, the expected loss becomes

$$E\left[l(e)\right] = E(ce^2) = c\sigma_p^2 \doteq cPQ/n \ . \tag{31}$$

The total costs which are affected by the size of the sample in this case would then be

$$C' = nC_s + cPQ/n \ . \tag{32}$$

To determine the optimum simple random sample for minimum total costs one can take the partial derivative of C' with respect to n and equate it to zero, from which n can be calculated. Hence, when

$$\frac{\partial C'}{\partial n} = C_s - \frac{cPQ}{n^2} = 0 , \tag{33}$$

$$n = \sqrt{\frac{cPQ}{C_s}} .$$

In this case, the optimal sample size turns out to be proportional to the square root of the cost ratio c/C_s and directly proportional to the standard deviation of a single observation.

In designing a sample to meet a stated cost, without considering the precision of the estimator, the above expression for total costs

$$C = C_1 + nC_s + cPQ/n \tag{34}$$

is solved for n to give

$$n = \frac{(C - C_1) \pm \sqrt{(C - C_1)^2 - rC_s cPQ}}{2C_s} .$$

This value of n may then be used in equation (21) to determine the precision which will be attained for this fixed cost, C. This calculation could lead to an abandonment of the idea of making the work sampling study if the resulting precision is not acceptable, and if the allocated resources cannot be increased. Likewise, the value of n calculated using equation (21) for a stated level of precision may be used in equation (34) to determine the cost of the stated precision. Obviously, one cannot attain an arbitrarily chosen level of precision and enjoy minimum costs simultaneously. A compromise must be worked out as indicated in the following section.

Selection of Sample Size Considering both Precision and Cost

If the costs associated with estimating each element are the same in the illustration presented earlier (Appendix A), and c/C_s is 2×10^7, say, then the sample sizes which would afford minimum cost for estimating each parameter are (using equation 33):

$$n_1 = 1660$$
$$n_2 = 1953$$
$$n_3 = 2210 .$$

The sample sizes which provided estimates within $\pm$ 10 per cent of the true values with 95 per cent confidence were

$$n_1 = 1944$$
$$n_2 = 1111$$
$$n_3 = 280 .$$

Simultaneous study of these sample sizes for a given study permits the evaluation of both cost and precision. The most important element(s) must be chosen for the purpose of determining the final sample size to be used. After an element is chosen, a choice must be made between realizing minimum costs, obtaining a desired precision, or some compromise between these two factors. For example, if the first element above is chosen as being the most important, a sample of 1944 will yield an estimate which is 95 per cent certain of being within 10 per cent of P_1. However a sample of 1660 will yield an estimate which will be optimum from the cost standpoint. Either one of these values of n or some intermediate value should be chosen, depending upon the desires of the decision maker.

It should be noted that the direct measurement of losses may not be feasible in many cases; however the current practice of work sampling in which the decision maker specifies the error tolerance and a level of confidence for the estimates, results in his placing a limit on $l(e)$ within which he wishes to operate. The decision of what the precision shall be in a given study should therefore be made with prudence in order to avoid the introduction of intolerable or unnecessary restrictions on the sample design. A reasonable approach to this decision when a loss function is not available is to consider jointly the expected costs of achieving different levels of precision, which can be measured, and the expected losses at each level, a judgment. The relationship between the sample size and level of precision is given for levels of confidence of 0.90, 0.95, and 0.99 in Figures 1, 2, and 3 of this chapter. While one cannot make an optimal decision in this case, he can observe the effects of greater precision on sampling costs and weigh this incremental cost against his judgment of losses due to errors in the estimates.

One other alternative is to formulate the problem in terms of mean deviations, which will avoid forcing the decision maker to rely on choosing a level of precision in terms of maximum deviations and confidence levels. In this case the decision maker is asked to supply an estimate of the average deviation of the estimator from the true value which he thinks should be achieved. Consequently, rather than having to specify a value of $|p-P|$ which he desires the error to be within at a chosen level of confidence, he is asked to estimate the average error, a somewhat simpler task. When this value is made available, it may be converted to its equivalent in terms of maximum variation and confidence level, which can then be used to determine the sample size, using Figures 1, 2, and 3. A joint use of these two approaches is likely to yield a more meaningful solution than either of them used separately.

The mean deviation of a variable which follows a normal distribution, as is assumed for p, is 0.7979 standard deviations from the mean. The expression for converting this to a value of R is given on each of the graphs in Figures 1, 2, and 3, as well as the expression for average error in terms of R and P. These expressions result from observing that the average deviation, denoted by $\bar{E}$, is

$$\bar{E} = 0.7979\ \sigma_p \tag{35}$$

and from equation (21),

$$RP = K_{\alpha/2}\ \sigma_p\ ,$$

from which

$$R = \frac{K_{\alpha/2}\ \bar{E}}{0.7979P}\ . \tag{36}$$

Limitations of Simple Random Work Sampling

If it were feasible to ignore questions of practicability and costs, simple random work sampling and its related theory would provide a solution to every work sampling problem. The desire to keep the theory simple and easy to understand has promoted the extensive use of simple random theory in situations where the simple random procedure has been deviated from in order to provide greater flexibility. For example, pq/n is commonly considered to be the variance of a proportion, p, regardless of the sampling scheme used in collecting the data. Simple random theory is not appropriate except where simple random sampling has been used in collecting the data, however if a known scheme of probability sampling is used, the proper estimation theory can be determined.

The remaining chapters of this study are devoted to an investigation of a variety of sampling schemes for use in work sampling studies. However, when simple random sampling, as presented in this chapter, is designated as the proper sampling scheme in a work sampling study, the limitations as herein stated must be observed. The problems of specifying the sampling unit and estimating population parameters for purposes of design may easily lead one to claim a precision not actually obtained. With respect to costs, it has been shown that arbitrary designation of estimator precision may lead to work sampling plans which are far from optimal. Careful considerations of costs as herein discussed will assist in obviating this problem.

CHAPTER IV

STRATIFIED RANDOM WORK SAMPLING

The Scope of this Chapter

With respect to the second major objective listed in Chapter I, survey sampling practice has shown that one of the simplest ways of increasing the efficiency of a simple random sampling plan is through the use of stratification. In stratification, supplemental information is used to divide the universe under study into groups of sampling units in which the characteristics of the universe under study are more homogeneous than for the universe as a whole. The groups formed are called strata, and when simple random samples are drawn from each group, the process is called stratified random sampling.

The theoretical consequences of sampling in the above described fashion is an increase in the precision of the estimators if the sample size is maintained; or more important, the same precision is gained with a smaller sample than is required in simple random sampling. There have been numerous applications of stratification in survey sampling which deal with the measurement of continuous characteristics; occasionally, the procedure has been utilized in the measurement of proportions, and hence is applicable in work sampling. This chapter is devoted to the development of a stratified random work sampling model and will consider the following problems:

(a) Selection of strata,

(b) Determining sample sizes for each stratum, including cost concepts as well as variance concepts,

(c) Selection of samples from each stratum,

(d) Preparation of estimates for the entire universe from the composite sample,

(e) Calculation of the error variance of the sample estimators.

Finally, a comparison of stratified random work sampling with simple random work sampling will be made in terms of estimator variances as well as overall costs. An illustration of stratified random work sampling will also be included.

The Nature of Stratification in Work Sampling

It has not been unusual for work sampling practitioners to select stratified samples. A common procedure has been to determine the number of observations needed for a stated level of precision (using the variance concepts from simple random work sampling) and then to divide the total number of observations evenly over the days of the study. In this case the observations are stratified with respect to days although simple random work sampling formulas are used in making confidence statements about the parameters being estimated. Obviously, this practice of stratification has been for convenience in data collection rather than for increasing the precision of the estimators, the theoretical reason for stratifying. Both the practical and the theoretical gains are possible in stratified random work sampling if the sample design is properly constructed and carried out. It turns out that in most cases, stratification on the basis of days will increase the precision of the estimators.

The gain in the precision of estimators with stratified random work sampling is determined by how well one is able to group the universe into strata which are more homogeneous than the universe as a whole. The grouping of units in this manner serves to divide the variance of the total universe into two components, the within strata variance and the between strata variance, i.e.,

$$\sigma^2_{Total} = \sigma^2_{Between} + \sigma^2_{Within} . \tag{37}$$

These variance components are shown more fully in equations (69) to (71). Subsequent estimates which are functions of samples selected from these strata reflect only the "within" component of variance, since the strata are sampled 100 per cent, i.e., observations are made in all of the strata. Thus the more unalike one makes the groups, the greater $\sigma^2_{Between}$ becomes, the smaller σ^2_{Within} becomes, and the more precise are the estimators. For example, if the universe could be separated into two strata in which one stratum consisted of all units possessing a characteristic, and the other stratum consisted of all units not possessing the characteristic, then an estimate of p, the proportion of the entire universe possessing the characteristic, could be made which has zero variance by taking a single observation.

It is important that the creation of strata utilizes all of the available knowledge pertaining to the structure of the universe. As a rule, the industrial engineer is well acquainted with the characteristics of an activity before he proceeds to study it whether the tools used include work sampling or not. Using the knowledge which he can make available, such as known differences in the characteristics of the activity over the time period the study will encompass, he can successfully stratify the activity for the purposes of sampling. It is important to note that stratification is a judgmental process but as long as probability samples are taken from the strata, this does not preclude the use of probability sampling theory. Due to the necessity for exercising judgment, it is not expected that "optimum" stratification will be gained in this manner. However, it will be seen later that the stratification process will yield more efficient sampling methods even though stratification is far from optimum. In fact, if the strata are formed by randomly assigning the units in the population to the strata, resulting in no significant differences among strata, and a proportional stratified sample is taken from each stratum (sample sizes proportional to strata sizes), estimators based on stratified random sampling will have essentially the same variance as those based on simple random sampling.

In determining whether or not stratified random work sampling should be used, as opposed to some other sampling scheme, the industrial engineer only needs to know whether or not subperiods of the overall activity can be formed which differ significantly with respect to the proportion of time spent in the states of the activity of interest. Even when such a characteristic of the process is not highly significant, stratification may still offer administrative advantages. This is clearly evident in current practice as cited at the beginning of this section. Consider the following common situations which could logically lead to stratification:

1. The worker spends the beginning portion of the day getting "warmed up" or set up for work and the latter portion of the day

cleaning up and "putting away." The proportion of time spent in the state of the activity, say, "operate machine" would not be the same for all periods of the day. Stratification could thus be made on hours or some other period, each hour or period being a unique stratum.

2. Absenteeism, time of week (Monday vs. Friday), weather, etc., may create differences in each days performance and cause the fraction of time spent in given states of the activity to vary significantly from day to day. Stratification by days is indicated. Any other subperiod of time, among which there are significant differences in the characteristics being measured, would also serve as a basis for stratification.

3. Differences in materials, types of orders, etc., could introduce differences in the time required for various elements from order to order. Hence stratification by orders could be profitable.

4. Individual differences in people or machines will introduce differences in the time required for performing the elements of an activity. Stratification with respect to one or more defined characteristics of people and/or machines would be possible.

Situations such as the foregoing are recognized as being quite common in practice. There is undoubtedly a number of other factors which would be peculiar to a specific activity under study and which would serve as bases for stratification. Practitioners in the field of survey sampling (31) have stated that populations seldom exist in which there is no characteristic on which one can stratify. This leads to the conclusion that simple random sampling should seldom be employed as the overall sampling procedure, but that it should serve instead as a useful basis for developing more efficient models. The following sections of this chapter will show the exact nature and consequences of applying stratification in work sampling studies.

Stratified Random Work Sampling

The sampling unit in stratified random work sampling is the same as defined in simple random work sampling. That is, the population is divided into minutes and when a minute is chosen for observation, the actual unit sampled is a random instant within that minute. The entire universe of U minutes, of N instants, is divided into L groups or strata and the number of sampling units in the hth stratum is N_h. That is, the units in the hth stratum are assumed to be numbered 1, 2, ..., N_h. Since there are L strata and each unit appears in one and only one stratum, $N = \sum_{h=1}^{L} N_h$. In a similar fashion, the sample size in the hth stratum is n_h and $n = \sum_{h=1}^{L} n_h$, where n is the size of the aggregate sample from all strata. The value of a characteristic, say X, of the ith unit from the hth stratum is X_{hi}. In work sampling, each element of the activity under study will be designated with a unique notation such as X, Y, or Z as in Chapter III. For the hth stratum, the value of X_{hi} represents the presence or absence of element X on the ith unit in the hth stratum; Y_{hi} represents the presence or absence of element Y on the ith unit; and Z_{hi} represents the presence or absence of element Z on the ith unit. Therefore, for each unit in the stratum, either X_{hi}, Y_{hi}, or Z_{hi} will be equal to one and the other two will be equal to zero if the only elements possible are X, Y, and Z. As an example, consider Appendix A. If the total scope of the study ($U = 19{,}200$ man-minutes) is divided into ten strata, where each day is a stratum, then each stratum contains a total of 1,920 man-minutes. This accounts for the total scope of time under consideration. There is assumed to be an infinite number of instants in the population as well as in each stratum. In the hypothetical activity in Appendix A, each man-minute is occupied by one, and only one, state of the activity. However, this does not need to be the case (as pointed out earlier under the discussion of the sampling unit in Chapter III) since a random instant within the minute chosen is what is actually sampled. It should also be noted that strata do not have to be equal in size for the application of sampling theory in developing estimators and their variances. This practice is followed here for convenience.

The total number of instants devoted to the Xth state of the activity in stratum h is

$$X_h = \sum_{i=1}^{N_h} X_{hi} \; . \tag{38}$$

Y_h and Z_h are determined analogously. Since the three states of the activity are mutually exclusive (Appendix A) as well as all inclusive, then $X_h + Y_h + Z_h = N_h$. It is not necessary that the elements of concern in a work sampling study be all inclusive; however if this is not the case, it is advisable to label a state as "other" to insure that a recording is made at each observation. Such practice will serve as a check on the total number of observations made and will also assist in keeping the results free of non-sampling errors. Following the notation above, and using Appendix A, x_{hj} will denote the presence or absence of state X at the jth sample observation in stratum h.

Hence $x_h = \sum_{j=1}^{N_h} x_{hj}$. The sums y_h and z_h are calculated analogously.

The total number of instants in the universe devoted to each element of an activity in which there are three elements, X, Y, and Z, is

$$X = \sum_{h=1}^{L} X_h = \sum_{h=1}^{L} \sum_{i=1}^{N_h} X_{hi} \tag{39}$$

$$Y = \sum_{h=1}^{L} Y_h = \sum_{h=1}^{L} \sum_{i=1}^{N_h} Y_{hi}$$

$$Z = \sum_{h=1}^{L} Z_h = \sum_{h=1}^{L} \sum_{i=1}^{N_h} Z_{hi} \ .$$

The total number of instants in the sample of size n at which the activity is observed to be in each of the three states is

$$x = \sum_{h=1}^{L} x_h = \sum_{h=1}^{L} \sum_{j=1}^{N_h} x_{hj} \tag{40}$$

$$y = \sum_{h=1}^{L} y_h = \sum_{h=1}^{L} \sum_{j=1}^{N_h} y_{hj}$$

$$z = \sum_{h=1}^{L} z_h = \sum_{h=1}^{L} \sum_{j=1}^{N_h} z_{jh} \ .$$

The proportions of time spent in each of the three states over the entire universe are

$$\bar{X} = P_X = X/N \tag{41}$$

$$\bar{Y} = P_Y = Y/N$$

$$\bar{Z} = P_Z = Z/N \ .$$

The proportions for the hth stratum are

$$\bar{X}_h = P_{Xh} = X_h/N_h \tag{42}$$

$$\bar{Y}_h = P_{Yh} = Y_h/N_h$$

$$\bar{Z}_h = P_{Zh} = Z_h/N_h \ ,$$

and for the sample of n_h items from the hth stratum, these proportions are

$$\bar{x}_h = p_{Xh} = x_h/n_h \tag{43}$$

$$\bar{y}_h = p_{Yh} = y_h/n_h$$

$$\bar{z}_h = p_{Zh} = z_h/n_h \ .$$

As in simple random sampling, the variance over the entire universe for each of the characteristics X, Y, and Z will be denoted by

$$S_X^2 = \frac{1}{N-1} \sum_{h=1}^{L} \sum_{i=1}^{N_h} (X_{hi} - \bar{X})^2 \ , \quad \text{etc.} \tag{44}$$

The variances of the characteristics within the hth stratum are denoted similarly by

$$S_{Xh}^2 = \frac{1}{N_{h-1}} \sum_{i=1}^{N_h} (X_{hi} - \bar{X}_h)^2, \quad \text{etc.} \tag{45}$$

The magnitude of the gains in precision from stratified random work sampling as compared to simple random work sampling will depend upon how effectively the stratification is carried out. Improvements in the methods of data collection depend upon the practitioners' ability to create strata which are easier to sample as a group than the population as a whole. Analyses of the gains in both cases will be made and discussed in later sections.

Estimators from a Stratified Random Sample

Since the population characteristics to be estimated in work sampling are proportions, this discussion will be restricted to this type of estimator. As indicated earlier, the mean of a population characteristic, say X, where the population has been divided into L strata, is

$$P_X = X/N = \frac{\sum_{h=1}^{L} N_h P_{Xh}}{\sum_{h=1}^{L} N_h} = \frac{1}{N} \sum_{h=1}^{L} N_h P_{Xh} \, . \tag{46}$$

Thus P_X is a weighted mean of the P_{Xh}'s, the proportions in the various strata. The weighting factor N_h/N is, as one would expect, the proportion of the population in each stratum.

Since a simple random sample is drawn from each stratum, Chapter III may be referred to for showing that

$$\bar{x}_h = p_{Xh} = x_h/n_h = \frac{1}{n_h} \sum_{j=1}^{n_h} x_{hj} \tag{47}$$

is a consistent and unbiased estimator of $\bar{X}_h = P_{Xh}$, the true mean of stratum h. Hence to obtain a consistent and unbiased* estimator of the overall proportion P_X, we use

$$p_X = \frac{1}{N} \sum_{h=1}^{L} N_h p_{Xh} \; . \tag{48}$$

If proportionate sampling is used, i.e., if the sampling fraction n_h/N_h is equal to n/N for each stratum and thus $N_h = Nn_h/n$, the estimator p_X reduces to

$$p_X = \frac{1}{N} \sum_{h=1}^{L} N_h p_{Xh} = \frac{1}{n} \sum_{h=1}^{L} x_h = \frac{1}{n} \sum_{h=1}^{L} \sum_{j=1}^{n_h} x_{hj}, \tag{49}$$

which yields the same estimate as simple random sampling. It is obvious that the calculation of the estimate is simplified considerably when proportionate sampling is used as compared to disproportionate sampling. The necessity of weighting each stratum estimate separately is eliminated in the former case. The practice cited earlier of using simple random

---

*See Cochran (15, p. 91).

sampling concepts but stratifying over days is an example of proportionate stratified sampling. Current practice has established that proportionate sampling has many administrative advantages of which the major one is the more even demand on the observer's time. Although the estimate obtained will be the same in proportionate stratified random work sampling as in simple random work sampling, the variance of the stratified estimator is smaller than the variance of the simple random estimator; and for this reason, the usual practice of ignoring stratification in computing the standard deviation of the estimate results in an overestimate. Hence a desired precision may be attained with a smaller sample using the stratified estimator. A comparison of these variances is made in a later section of this chapter. An investigation of proportionate versus disproportionate sampling in work sampling will also be made later.

Precision of the Estimators

Using the variance of a simple random sample from Chapter III, i.e.,

$$\sigma^2_{p_X} = \left(\frac{N-n}{N-1}\right) \frac{P_X Q_X}{n} , \tag{50}$$

we see that the variance of p_{Xh}, the estimated proportion for the hth stratum is

$$\sigma^2_{p_{Xh}} = \left(\frac{N_h-n_h}{N_h-1}\right) \frac{P_{Xh} Q_{Xh}}{n_h} . \tag{51}$$

Using two well known mathematical theorems, (1) that the variance of a sum of independent components is equal to the sum of the variances of the components, and (2) that the variance of a constant times a variable is equal to the constant squared times the variance of the variable, we see that, for a stratified sample,

$$\sigma^2_{p_X} = \sum_{h=1}^{L} \left(\frac{N_h}{N}\right)^2 \sigma^2_{p_{Xh}} = \sum_{h=1}^{L} \left(\frac{N_h}{N}\right)^2 \left(\frac{N_h-n_h}{N_h-1}\right)\left(\frac{P_{Xh}Q_{Xh}}{n_h}\right). \tag{52}$$

This is the variance of p_X from a finite population of N units. Since the practice of sampling a random instant within the chosen minute leads to the assumption of an infinite population of "ultimate" sampling units, this variance reduces to

$$\sigma^2_{p_X} = \sum_{h=1}^{L} \frac{N_h^2 P_{Xh} Q_{Xh}}{N^2 n_h} = \sum_{h=1}^{L} \frac{f_h^2 P_{Xh} Q_{Xh}}{n_h} , \tag{53}$$

where $f_h = N_h/N$ is the fraction of the population in the hth stratum. Further, in the case of proportional sampling, where $n_h/N_h = n/N$, this variance reduces to

$$\sigma^2_{p_X} = \sum_{h=1}^{L} \frac{N_h^2 P_{Xh} Q_{Xh}}{N n N_h} = \frac{1}{n} \sum_{h=1}^{L} f_h P_{Xh} Q_{Xh} . \tag{54}$$

As in the case of simple random sampling, the above variances contain the parameters being estimated and hence will have to be estimated from the sample itself. Applying the procedure for estimating variances from the sample given in Chapter III to each stratum we see that

$$s^2_{p_{Xh}} = \left(\frac{N_h-n_h}{N_h-1}\right) \frac{p_{Xh} q_{Xh}}{n_h} \tag{55}$$

is an unbiased (and consistent) estimator of $\sigma^2_{p_{Xh}}$. Therefore, we substitute $s^2_{p_{Xh}}$ in equation (52) and obtain

$$s^2_{p_X} = \sum_{h=1}^{L} \left(\frac{N_h}{N}\right)^2 s^2_{p_{Xh}} = \sum_{h=1}^{L} \left(\frac{N_h}{N}\right)^2 \left(\frac{N_h - n_h}{N_h - 1}\right) \frac{p_{Xh} q_{Xh}}{n_h} , \qquad (56)$$

and upon assuming an infinite population and proportionate stratified sampling, the estimated variance becomes

$$s^2_{p_X} = \sum_{h=1}^{L} \frac{N_h p_{Xh} q_{Xh}}{nN} = \frac{1}{n} \sum_{h=1}^{L} f_h p_{Xh} q_{Xh} . \qquad (57)$$

Designing a Stratified Random Sample for a Stated Precision

As indicated in the general comments made earlier about stratification in work sampling, the objective in setting up the strata is to make them as homogeneous as possible within themselves, and as different as possible from stratum to stratum. The number of strata formed will reflect the depth to which one stratifies a population. From the standpoint of variance reduction in the estimators, it is theoretically possible to continue to make gains by making more and more and consequently smaller and smaller strata. However, the increased administrative detail in such cases soon offsets the additional gain from deeper stratification. This phenomenon is more significant with zero-one variables as used in work sampling, for if the practitioner is unable to show significant differences in the values of P from stratum to stratum, additional stratification is not profitable. Major factors such as those discussed earlier (periods, individuals, orders, etc.) will undoubtedly yield most of the gains in stratified work sampling. As stated in that discussion, the choice of factors on which to stratify must be left to the judgment of the practitioner.

In some cases the factors on which to stratify may not be clear due to a lack of knowledge about the values of the P_h's for subsets of the population. One may take a preliminary sample for the purpose of getting rough estimates of the P_h's to use in the design of the sample as well as to obtain a knowledge of the best factors for stratification. However, in cases where there have been previous studies, data obtained in these studies may be used to indicate the answers to these questions.

When the decision is made as to the method for specifying the strata and an estimate of the P_h's is secured, the total sample size for a stated level of precision of the estimator can be estimated. Limitations on the magnitude of the sample may take two forms as in simple random work sampling. A desired precision may be specified or a total cost for the study may be the limiting factor. The first of these will be considered here, the latter will be discussed in the section which pertains to costs in stratified random work sampling.

Using the same general notation of Chapter III, the estimate of P_X will be p_X as stated in equation (48). The variance of this estimator is given by equation (54), where proportionate sampling is assumed. The subscript X is dropped in this derivation for clarity. It is to be understood that the calculations are for a particular parameter. If the estimate is desired within a fraction, R, of the true value with confidence $1 - \alpha$, then

$$RP = K_{\alpha/2}\sigma_p = K_{\alpha/2}\left\{\frac{1}{n}\sum_{h=1}^{L} f_h P_h Q_h\right\}^{1/2} \tag{58}$$

$$n = \frac{K^2_{\alpha/2}\sum_{h=1}^{L} f_h P_h Q_h}{R^2 P^2} .$$

This is the value of n necessary to yield an estimate of P that will be within 100R per cent of the true value with confidence $1-\alpha$.

Again it is necessary to have rough estimates of the P_h's in order to use this formula. These may be approximated from existing knowledge of the population or from pilot studies which are made for this purpose or for the purpose of defining the strata.

The variance of $s^2_{p_X}$ is a complicated expression; however the rules in Chapter III for assuring that $V_s \leq 0.10$ may be applied to each stratum to insure that this will not be a source of significant error in the estimated confidence limits on P_X. As before, the fact that sampling is assumed to be from an infinite population in work sampling and the fact that the estimated proportions are not often less than 0.10, the sample will almost always be sufficient to allow the assumption that $s^2_{p_X} = \sigma^2_{p_X}$.

Following the same general procedure set forth in Chapter III, but using the variance of the stratified estimator, one can set confidence interval estimates on the proportion of time spent performing the ith element as follows:

$$P(p_i - k_{\alpha/2}s_{p_i} \leq P_i \leq p_i + k_{\alpha/2}s_{p_i}) \doteq 1-\alpha = \int_{-K_{\alpha/2}}^{+K_{\alpha/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} dr. \tag{59}$$

Such intervals are stated under the assumption that p_i is a normal variable with standard deviation s_{p_i}, hence the approximation.

Allocation of the Sample, n, to the Strata in Stratified Work Sampling

After the total sample size has been determined, the question of how the sample will be allocated to the strata remains to be answered. In fact, this information must be known before the sample size is calculated since the variance of the estimator depends on this allocation. This problem has received considerable attention in survey sampling practices and the solutions there offer some insight into how this should be accomplished in work sampling (15, p. 73). In the prior sections of this chapter, expressions for estimators and their variances were given for proportional allocation of the sample to the strata. This procedure simply specifies that the sample in each stratum, n_h, will be $(n/N)N_h$.

In estimating a proportion, P, using a stratified random sample from a finite population, it was shown that p (equation 48) is a consistent and unbiased estimator of P with variance as shown in equation (52), which reduced to $\sum_{h=1}^{L} f_h^2 P_h Q_h / n_h$ in stratified work sampling (equation 53). One objective in specifying the n_h for each stratum is to minimize the variance of the estimator for a given total sample size n (this practice ignores costs of sampling which will be treated later). The magnitude of n may be the result of imposed limits on the precision of the estimator or imposed limits on total costs. The former was discussed in the preceding section; the latter will be discussed in the section on costs.

Considering the problem of determining the allocation of n to the strata, the objective is to minimize the variance of the estimator subject to the condition that $\sum_{h=1}^{L} n_h = n$. Following the technique advanced by Cochran (15, p. 73), the Lagrange multiplier is used to select n_h to minimize

$$\sigma_p^2 + \lambda\left(\sum_{h=1}^{L} n_h - n\right) \tag{60}$$

$$= \sum_{h=1}^{L} f_h^2 \frac{P_h Q_h}{n_h} + \lambda\left(\sum_{h=1}^{L} n_h - n\right) .$$

Differentiating with respect to n_h, one obtains

$$\frac{-f_h^2 P_h Q_h}{n_h^2} + \lambda = 0$$

$$n_h = \left(\frac{f_h^2 P_h Q_h}{\lambda}\right)^{1/2} = \frac{1}{\lambda^{1/2}} (f_h^2 P_h Q_h)^{1/2} .$$

$$\sum_{h=1}^{L} n_h = n = \frac{1}{\lambda^{1/2}} \sum_{h=1}^{L} f_h (P_h Q_h)^{1/2}$$

and

$$\lambda^{1/2} = \frac{1}{n} \sum_{h=1}^{L} f_h (P_h Q_h)^{1/2} .$$

Substituting the expression for $\lambda^{1/2}$ into the expression for n_h, we get

$$n_h = \frac{nf_h(P_hQ_h)^{1/2}}{\sum_{h=1}^{L} f_h(P_hQ_h)^{1/2}} \quad . \tag{61}$$

This result shows that the optimum allocation of a fixed sample, n, for the purpose of minimizing the variance of the estimator is to make the allocation a function of the relative sizes of the strata, f_h, and the stratum standard deviations, $(P_hQ_h)^{1/2}$. In essence, this procedure says to take a larger portion of the total sample, n, from the larger and more variable strata.

Since P ranges from 0 to 1 in all cases, the general gain for minimum variance allocation over proportional allocation can be indicated. The variance of the estimator of P for proportional allocation is given in equation (54) and can be rewritten as $\sum_{h=1}^{L} f_hP_hQ_h/n$, whereas in minimum variance allocation, this variance is found by substituting the value of n_h calculated above into equation (53), which becomes

$$\sigma_p^2 = \sum_{h=1}^{L} \frac{f_h^2\,P_hQ_h}{n} \frac{\sum_{h=1}^{L} f_h(P_hQ_h)^{1/2}}{f_h(P_hQ_h)^{1/2}} \tag{62}$$

$$= \sum_{h=1}^{L} \frac{f_h(P_hQ_h)^{1/2}}{n} \sum_{h=1}^{L} f_h(P_hQ_h)^{1/2}$$

$$= \frac{\left(\sum_{h=1}^{L} f_h\sqrt{P_hQ_h}\right)^2}{n} \quad .$$

Hence, the ratio of the variances for minimum variance allocation to proportional allocation is

$$\frac{\sigma_p^2 \text{ (Minimum variance)}}{\sigma_p^2 \text{ (Proportional)}} = \frac{\left(\sum_{h=1}^{L} f_h \sqrt{P_h Q_h}\right)^2}{\sum_{h=1}^{L} f_h P_h Q_h} . \tag{63}$$

Although the possible values of this ratio are great in number due to the many values possible for L, f_h, and P_h, an evaluation of its magnitude can be given in a typical work sampling setting. It is logical that in many work sampling studies where stratification would be administratively convenient, the L strata will be equal in size. For example, stratification by days, as pointed out earlier to be a common practice, yields strata of equal size. Stratification by individual subjects would also yield strata of equal size. The ratio of minimum variance to proportional allocation for strata of equal size $(f_h = 1/L)$ is

$$\frac{\left(\sum_{h=1}^{L} \sqrt{P_h Q_h}\right)^2}{L \sum_{h=1}^{L} P_h Q_h} . \tag{64}$$

Cochran shows that for $0.5 \leq P_h \leq 0.95$ and for $h = 2$, that the relative precision of proportional to minimum variance allocation for the most severe cases of P_h is approximately 87 per cent (15, pp. 91-92). He further states that "In most cases the simplicity and the self-weighting feature of proportional stratification more than compensate for this slight loss in precision."

In addition to this indication that proportional allocation is preferred to minimum variance allocation, an allocation based on costs of sampling in each stratum will shed additional light on the problem. The total cost of sampling in the hth stratum is $n_h c_h$, and total sampling costs are $\sum_{h=1}^{L} n_h c_h$. For the sample of size n as calculated earlier, the allocation is made such that the variance of the estimator is minimized subject to a specified sampling cost, say C. The procedure used earlier in minimizing the variance of the estimator subject to a fixed sample size leads to minimizing the same variance subject to a fixed cost, i.e.,

$$\sigma_p^2 + \lambda\left\{\sum_{h=1}^{L} n_h c_h\right\} = \sum_{h=1}^{L} f_h^2 \frac{P_h Q_h}{n_h} + \lambda \sum_{h=1}^{L} n_h c_h . \tag{65}$$

Differentiating with respect to n_h and equating the result to zero yields

$$-\frac{f_h^2 P_h Q_h}{n_h^2} + \lambda c_h = 0 ,$$

and solving for n_h, one obtains

$$n_h = \frac{f_h (P_h Q_h)^{1/2}}{(\lambda c_h)^{1/2}} . \tag{66}$$

But since $n = \sum_{h=1}^{L} n_h$, then

$$n = \sum_{h=1}^{L} \frac{f_h (P_h Q_h)^{1/2}}{(\lambda c_h)^{1/2}} , \quad \text{and} \tag{67}$$

$$n_h = \frac{nf_h\left(\frac{P_hQ_h}{c_h}\right)^{1/2}}{\sum_{h=1}^{L} f_h\left(\frac{P_hQ_h}{c_h}\right)^{1/2}} .$$

This procedure allocates n to the L strata in such a way that larger samples are taken in strata where the variance is greater, where the size is larger, and where the cost of sampling is less.

Since costs affect allocation in proportion to the square root of the per unit cost within each stratum, cost differences between strata have little influence in the allocation unless they are as much as a factor of three or four. In other words, this method of allocation is not very sensitive to small differences in cost.

It is not often that the costs of sampling between strata in stratified work sampling will be different at all, to say nothing of being of the magnitude necessary for significance. It costs essentially the same to make an observation on any day, on any individual, machine, etc. In such cases, allocation as a function of sampling costs within strata does not yield significant gains over proportional allocation.

The foregoing analyses of allocation to strata as functions of sample size and costs are in terms of a single estimator. That is, the best method of allocation was determined when a single parameter was being estimated. In work sampling, it is usually the case that estimates of time spent in several categories of the activity are desired. In that case, when the allocation is optimized with respect to each of the estimators, the allocations will be in conflict. Hence, if one is interested in more than one estimator, some compromise among the optimum allocations must be reached. It was noted earlier in comparing minimum variance allocation with proportional allocation that the relative precision is close to one, even when the P_h's are drastically different. In the light of this fact, and the conflicting optimizations when several estimates are desired, it appears that proportional allocation will be the best compromise. The administrative advantages of proportional allocation cited earlier add to the desirability of this choice. The estimators and their variances for proportional allocation were stated in an earlier section.

<u>A Variance Comparison of Proportional Stratified Random Work Sampling and Simple Random Work Sampling</u>

It was stated earlier that the gains made in reducing the variance of an estimator by using stratification is a function of how different the strata are made. This is reflected in a statement made earlier in this chapter that the total variance in the universe could be, and is, divided into two components in the stratification process, viz., the variance between strata and the variance within strata. The magnitude of these components of variance will now be investigated in order to indicate quantitatively the gains in variance reduction when stratification is used in lieu of simple random sampling. The comparison will be between proportional stratified random work sampling and simple random work sampling.

The variance of an estimator, p, from a simple random work sample of size n is PQ/n (equation 19), whereas the variance of the estimator from a proportionally allocated stratified random work sample is $\sum_{h=1}^{L} f_h P_h Q_h / n$ (equation 54). Using standard analysis of variance notation, the variance of p_X from the simple random sample may be written in terms of the stratified population as

$$\frac{PQ}{n} = \sum_{h=1}^{L} \sum_{i=1}^{N_h} \frac{(X_{hi} - \bar{X})^2}{n(N - 1)} , \quad \text{or} \tag{68}$$

$$(N-1)PQ = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (X_{hi} - \bar{X})^2 .$$

Using the algebraic identity $(X_{hi} - \bar{X})^2 = \left[(X_{hi} - \bar{X}_h) + (\bar{X}_h - \bar{X})\right]^2$, in which the cross product term on the right vanishes, the right side of equation (68) may further be written as the sum of the within strata and between strata sums of squares, as follows:

$$(N-1)PQ = \sum_{h=1}^{L} \sum_{i=1}^{N_h} (X_{hi} - \bar{X}_h)^2 + \sum_{h=1}^{L} N_h(\bar{X}_h - \bar{X})^2 \tag{69}$$

$$= \sum_{h=1}^{L} (N_h - 1)P_h Q_h + \sum_{h=1}^{L} N_h(\bar{X}_h - \bar{X})^2 .$$

The terms $N-1$ and N_h-1 may be replaced by N and N_h respectively, since we are dealing with infinite populations, and division by nN yields

$$\frac{PQ}{n} = \frac{1}{n} \sum_{h=1}^{L} f_h P_h Q_h + \frac{1}{n} \sum_{h=1}^{L} f_h(\bar{X}_h - \bar{X})^2 , \tag{70}$$

which shows that

$$\sigma^2_{p_X} \text{ (random)} = \sigma^2_{p_X} \begin{Bmatrix} \text{stratified} \\ \text{proportional} \end{Bmatrix} + \frac{1}{n} \sum_{h=1}^{L} f_h(\bar{X}_h - \bar{X})^2 . \qquad (71)$$

The last term in this expression is a measure of the reduction in the variance of the estimator when simple random sampling is replaced by proportional stratified sampling. The term may be written in terms of the P_h's as follows:

$$\text{Reduction in variance} = \frac{1}{n} \sum_{h=1}^{L} f_h(P_h - P)^2 . \qquad (72)$$

It now becomes clear that the greater the differences in the P_h's, the greater the gain from stratification. In case all the P_h's are identical, the term vanishes and stratification is equivalent to simple random sampling.

<u>An Illustration of Stratified Random Work Sampling</u>

To illustrate proportional stratified random work sampling, the activity in Appendix A will again be considered. The factors which are most evident for purposes of stratification are those of time and subjects. Depending on the nature of the activity, a single factor will generally be most desirable for the purpose of stratification. One might consider stratification by individuals in the case of the activity in Appendix A. In this case, there would be four strata, each consisting of one-fourth of the total population. Or there might be reason to believe that variations from one time period to another are more significant than variations among individuals, and stratification over time would be better. In order to illustrate the principles of stratified random work sampling, the activity will be stratified by days, resulting in ten strata of equal size.

This illustration will be concerned with designing a stratified sample to yield estimates which are within a stated range of the true values at a desired level of confidence. In order to make a comparison with simple random work sampling, the same precision that was used in the illustration in Chapter III will be chosen. The precision there was that the estimates be within 10 per cent of the true values with a confidence of 95 per cent. There is need to estimate the P_i's for each of the ten strata in order to calculate the overall sample size, n. In this theoretical study, the actual values as given in Table 2 will be used. Using equation (58), which assumes proportional sampling, the values of n_1, n_2, and n_3 are calculated and are as follows:

$$\begin{aligned} n_1 &= 1464 \\ n_2 &= 544 \\ n_3 &= 161 \,. \end{aligned} \tag{73}$$

The sample sizes for simple random work sampling were

$$\begin{aligned} n_1 &= 1944 \\ n_2 &= 1111 \\ n_3 &= 280 \,. \end{aligned}$$

The latter sample sizes represent 32.8, 104.2, and 73.9 per cent more sampling than those for stratified random sampling with proportional

Table 2. Daily P_i's for the Activity in Appendix A

Day (Stratum)	P_1	P_2	P_3
1	.05	.08	.87
2	.08	.12	.80
3	.70	.20	.10
4	.10	.05	.85
5	.02	.85	.13
6	.20	.06	.74
7	.15	.14	.71
8	.12	.10	.78
9	.06	.90	.04
10	.18	.07	.75

allocation. It should be noted that the stratum differences in the P_i's represented in Table 2 may not be typical of industrial applications, since they were chosen such that the illustration would be significant. As already stated, the theoretical gain from stratification depends upon the ability of the practitioner to formulate homogeneous strata with significant differences between them, and the gain depends entirely on this fact. However, administrative gains may be realized even though the error variance reduction due to stratification is not highly significant.

As in the case of simple random work sampling, a compromise among the values of n_i for the various elements under study must be made. We choose the sample size corresponding to P_1 in Chapter III, hence it will be chosen here $(n = 1464)$ and the higher levels of precision will be stated for the estimates of P_2 and P_3 which this sample provides. Instead of $R = 0.10$, these levels of precision will be $R_2 = 0.061$ and $R_3 = 0.033$. These are calculated by solving equation (58) for R, using $n = 1464$.

The procedure for collecting the sample and analyzing the data for the stratified sample is as follows:

1. Since proportional allocation is made in each stratum, $1464/10 \doteq 146$ random observations are designated to each stratum (day) by the use of random numbers using the same procedure set forth in the case of simple random work sampling.

2. After the data are collected, equations (49) and (57) are used to calculate the estimates p_1, p_2, and p_3 and their variances.

3. Confidence interval estimates are made for the P_i's as discussed previously.

Figure 6 shows the results of 200 stratified random samples of 1464 each from the activity in Appendix A. The close agreement of the empirical data with the theoretical predictions is indicated alongside each distribution, where the actual parameters and the true variances of the estimators are compared with the estimated values. The gain in the precision of the estimators from stratified random work sampling is apparent since the stratified estimator for P_1 has the same variance as the simple random estimator but with a sample of 1464 as compared to 1944, and the variances of the estimators for P_2 and P_3 are less using stratified sampling, even though the smaller sample size is employed.

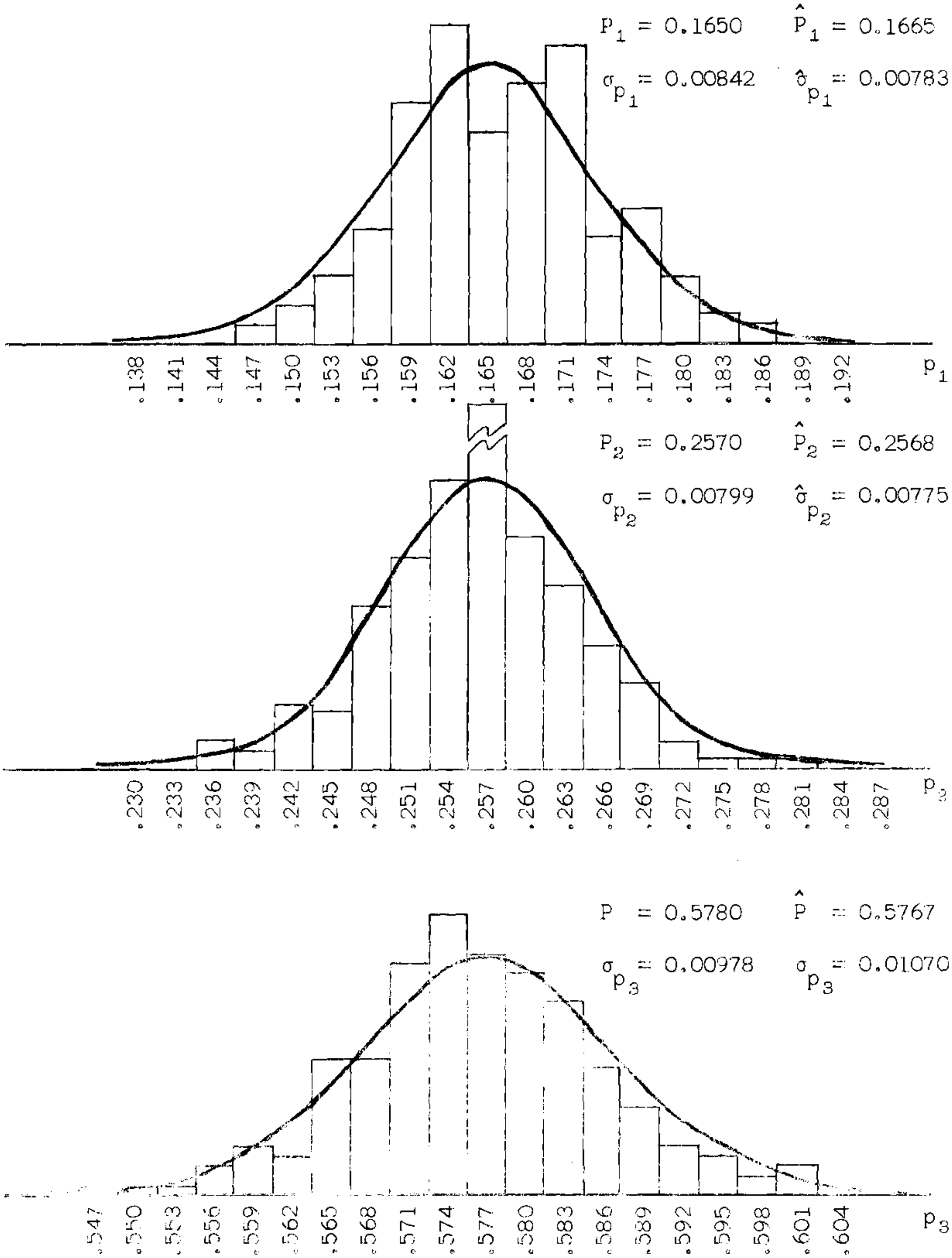


Figure 6. Frequency Distributions of p_1, p_2, and p_3 for 200 Studies Compared with Normal

Cost Factors in Stratified Random Work Sampling

The major components of work sampling costs which were listed in Chapter III are applicable regardless of the method of sampling used. The discussion on allocation revealed that the nature of cost differences between simple random sampling and stratified random sampling is usually with regard to the actual collection of the sample, i.e., sampling costs. The components of cost listed for simple random work sampling were

(1) Fixed costs

(2) Sampling costs

(3) Losses due to error.

The fixed costs and losses due to a stated error are not susceptible to changes due to different sampling procedures. The total sampling costs for simple random work sampling were nC_s, where n is the total number of units in the sample, and C_s is the cost of a single unit being included in the sample. In stratified sampling, the sampling costs are represented by the sum of the costs in the various strata, i.e., $\sum_{h=1}^{L} n_h c_h$, where n_h is the sample size and c_h is the unit cost of sampling in the hth stratum.

As revealed in the discussion on allocation, sampling costs in stratified random work sampling are not likely to be different from those in simple random work sampling when considered on a per unit basis. The strata which are formulated in work sampling (days, individual workers, orders, etc.) do not tend to be different in terms of the cost of drawing a unit into the sample. However, if these costs should be significantly different, they are easily reflected in the cost model with the expression for total sampling costs given above.

There is a possibility for increased costs in stratified random work sampling over simple random work sampling in the sample design and data analysis stages. In determining the stratification to be used, and the total sample size as well as its allocation to the strata, the demands on the practitioner's time are somewhat greater than in the case of the similar actions in simple random work sampling. It is also obvious that a minor increase in tabulation costs may occur with stratified sampling plans due to the calculations which must be made within each stratum. The magnitude of these increased costs is not readily calculable, however, and there is little justification for assuming that they will be highly significant. This is especially true with proportional allocation since this method of allocation does not introduce any complicated computational procedures.

To offset the foregoing tendencies for increased administrative costs in stratified work sampling studies over simple random studies, there are administrative gains to be expected from stratification. In efforts to stabilize the demands on the data collector's time, work sampling practitioners commonly divide the total sample in simple random work sampling studies over the period of the study such that the data collector makes about the same number of observations on the activity each day. Other objectives for this practice include the desire to have the data segregated by day, worker, etc., in order to observe differences in certain characteristics of the activity from day to day or from worker to worker. Again, the monetary benefits from such gains are not easily determined. It appears that the foregoing administrative aspects of costs are of such a nature as to be offsetting. Such an assumption will be made in the development of a cost model for stratified random work sampling.

<u>Designing a Stratified Random Work Sample to Minimize Costs or to Achieve a Stated Total Cost</u>

The determination of an optimal sample size based on minimizing the overall costs when stratified random work sampling is employed follows the same approach used in the case of simple random work sampling. In the latter case, when the "loss due to error" function of $l(e) = ce^2$ was used, it was found that $E\left[l(e)\right] = cPQ/n$. This loss function yields an expected "loss due to error" which is simply a constant times the variance of the estimator. In the case of stratified random work sampling with proportional allocation, this variance is given by equation (54) and is

$$\sigma_p^2 = \frac{1}{n} \sum_{h=1}^{L} f_h P_h Q_h .$$

The "loss due to error," $E\left[l(e)\right]$, is therefore

$$\frac{c}{n} \sum_{h=1}^{L} f_h P_h Q_h$$

for stratified random work sampling. With the costs of sampling in each stratum the same, the total cost equation for stratified random work sampling may be expressed as follows:

$$\text{Total Cost} = C_1 + \sum_{h=1}^{L} n_h c_h + C_e \tag{74}$$

$$= C_1 + nC_s + \frac{c}{n} \sum_{h=1}^{L} f_h P_h Q_h \;.$$

The administrative gains and losses are considered to be offsetting in this expression.

Hence, when the criterion for efficient design is that total costs shall be minimized, the above equation for total costs is minimized and solved for n as follows:

$$\frac{\partial Tc}{\partial n} = C_s - \frac{c}{n^2} \sum_{h=1}^{L} f_h P_h Q_h = 0 \tag{75}$$

$$n = \left\{ \frac{c}{C_s} \sum_{h=1}^{L} f_h P_h Q_h \right\}^{1/2} .$$

As in the case of designing the sample to yield a stated precision, the use of this procedure will require estimates of the various parameters involved prior to the study.

If a stated cost, C, is designated for the study, the sample size is determined by merely solving the foregoing total cost equation for n. The result is

$$n = \frac{(C-C_1) \pm \left\{ (C-C_1)^2 - 4C_s \, c \sum_{h=1}^{L} f_h P_h Q_h \right\}}{2C_s} \;. \tag{76}$$

The value of n derived in either of these cases may be used in equation (58) to determine the level of precision which will be attained for this fixed cost. Likewise, the total cost of a study which provides a stated precision may be determined by substituting the value of n calculated from equation (58) into the total cost equation above. The necessity of compromise between the two factors, variance and cost, will be considered in the next section.

Selection of Sample Size Considering both Precision and Cost

Equation (75) may be used to determine the optimal sample sizes for providing a minimum cost estimate of each of the parameters being estimated. Under the assumptions made in Chapter III, viz., that $c/C_s = 2 \times 10^7$, these sample sizes are

$$\begin{aligned} n_1 &= 1440 \\ n_2 &= 1368 \\ n_3 &= 1673 \,. \end{aligned} \tag{77}$$

These are to be compared with the sample sizes which will yield estimators with a desired variance. These were

$$\begin{aligned} n_1 &= 1464 \\ n_2 &= 544 \\ n_3 &= 161 \,. \end{aligned}$$

A single value must be chosen for the sample size and will reflect the decision-maker's choice of which element(s) are most important and whether desired variance or minimum cost is most important. This must necessarily be a judgmental decision, but can be made more rationally by knowing the consequences of the alternatives provided by the foregoing analysis.

<u>Limitations and Ramifications of Stratified Random Work Sampling</u>

As already indicated, if information is not available about the population for use in stratification, the costs of obtaining such information could possibly be more than the gains over simple random work sampling. While this is seldom believed to be a problem, if present, it would preclude stratifying the population. When pilot samples are needed for estimating the P_i's in order to be able to estimate sample sizes, knowledge for stratification may also be gained at little or no additional cost.

Since simple random sampling is used within each stratum, the limitations of simple random sampling from Chapter III apply to each stratum. Also, unless there are significant differences among the strata, the gains from stratification are expected to be small. If stratification is already being carried out, however, the gains from its use may as well be captured by using stratified estimators rather than simple random estimators. The gains will usually be positive, though perhaps small in such cases.

As pointed out in the section on allocation, when several variables are under study from the same sample, optimum allocation is not possible for all the estimators. There is no comprehensive theory available for making allocations in this case, and in work sampling, proportional allocation appears to give a procedure compatible with the various objectives of such a study.

Stratification will generally offer a gain in work sampling studies due to the fact that it can be introduced at no additional cost. There are certain other population characteristics to consider in designing a work sampling study. These will be considered in later chapters.

CHAPTER V

CLUSTER WORK SAMPLING

The Scope of this Chapter

In the models for simple random and stratified random work sampling, the observations made on the activity consisted of single determinations of the state of the activity at random points in time. Such observations are made of an individual worker, machine, or some other unilateral classification of the activity. In accomplishing the third specific objective stated in Chapter I, this chapter considers a model for work sampling wherein it is advantageous to observe a multilateral classification of activities such as two or more men, machines, etc., engaged simultaneously. The objective of this model is the same as for the previous ones -- to provide a desired amount of information for a minimum expenditure of resources. The measures sought are estimates of the proportions of time which the aggregate of subjects spend in predetermined states of the activity. The sampling procedure is developed for analyses where groups of constant as well as variable sizes, called clusters, are observed for simple random as well as for stratified random sampling of the groups. Comparisons of the efficiency of this method of sampling are made with simple random sampling based on the variances of the estimators as well as on total overall costs of the study.

The Nature of Clusters in Work Sampling

It has long been standard practice (16) for work sampling practitioners to take advantage of multiple observations (clusters) in work sampling. The practice commonly employed assumes that observations made in clusters are independent. This assumption then allows the application of approximate simple random work sampling theory. As pointed out in the brief synopsis of cluster sampling in Chapter I, this practice is likely to introduce gross errors into the error analysis. When a group of S workers, machines, etc., is observed, and S observations are made as to the state of the activity each member of the group is engaged in, the precision of the estimators from such samples is strongly dependent upon the nature of the interdependencies in the group. If the individual activities are even slightly correlated, which is often the case in human endeavor, then the variance of the estimator for P_i used in simple random and stratified random work sampling is considerably smaller than the variance of the estimator in cluster sampling.

There are numerous situations in work sampling in which cluster sampling will apply. A large class of these situations can be depicted by the hypothetical activity in Appendix A. If a random time observation is made on one of the members of the group of four workers shown there, it will require little additional effort to also make observations on the other members of the group. However, since the activities of the four workers in Appendix A are not independent, simple random work sampling theory does not apply in the analysis of the results. Actual situations such as this are not uncommon in practice, and a sampling model which would properly consider the interrelationships of the group would provide a realistic analysis of work sampling data obtained in such cases. Such a model would allow the practitioner to take advantage of the extra information gained by multiple observations without the limitations which vitiate the model now assumed. Additional information gained in multiple observations is seldom equivalent to taking a simple random sample of the same size, although this is what is frequently assumed.

Simple Cluster Work Sampling

The population subjected to sampling in the case of clustering is still the aggregate time interval which includes all of the subjects engaged in the activity. The population may still be regarded as an aggregate of U minutes, or N instants, where each point in time has a cluster of S instants associated with it. Whereas in stratified random work sampling the elementary units were grouped in homogeneous strata in order to reduce the variance of the estimator, the elementary units in cluster work sampling are observed in convenient groups for the purpose of reducing total sampling costs. The clusters of instants at a point in time are formed in a natural fashion by the S subjects engaged in the activity. The clusters will be referred to as primary sampling units (psu) since they will be sampled as groups and individual instants in the clusters included in the sample will be completely enumerated. The case of sampling the instants in the chosen clusters, rather than complete enumeration, is considered in the next chapter as a multi-stage sample.

In simple cluster work sampling, the sampling of the psu's will be by simple random sampling and will be subject to the following notation:

N = Number of elementary units (instants) in the population.

M = Number of psu's (clusters) into which the population is naturally grouped.

S_i = Number of elementary units (instants in the ith psu).

m = Number of psu's included in the sample.

$n = \sum_{i=1}^{m} S_i$ = Total number of elementary units (instants) in the sample.

As in previous chapters, the elements of the activity under study will be designated by letters such as X, Y, and Z. These will be zero-one variables and X_{ij} is the value of the Xth characteristic associated with the jth member of the ith psu. Each point in time will have one state of the activity associated with each member of the cluster at that point. Therefore, one of the variables will take on the value of one for each member of the cluster, and the remaining variables will take on the value zero. Since the entire cluster is included in the sample, x_{ij}, y_{ij}, and z_{ij}, the sample observations in the ith psu, will be equal to the population values, X_{ij}, Y_{ij}, and Z_{ij}.

The following notation, illustrated in terms of the element X, will also be necessary:

$$X_i = \sum_{j=1}^{S_i} X_{ij} = \text{Total number of elementary units in the ith}$$

psu which possesses characteristic X.

$$X = \sum_{i=1}^{M} X_i = \text{Total number of elementary units in the entire popu-}$$

lation which possesses the characteristic X.

$\overline{X} = X/M$ = Average number of elementary units per psu possessing characteristic X.

$\overline{\overline{X}} = X/N = P$ = Proportion of elementary units in the ith psu possessing characteristic X.

$\overline{\overline{X}} = X_i/S_i = P_i$ = Proportion of elementary units in the ith psu possessing characteristic X.

$$x = \sum_{i=1}^{M} X_i = \text{Total number of elementary units in a sample of } m$$

psu's which possess characteristic X.

$\overline{\overline{x}} = x/n = p$ = proportion of elementary units in the sample possessing characteristic X.

Estimators in Simple Cluster Work Sampling

The objective is to estimate P, the fraction of time the activity is in a given state. The main measure of effectiveness in cluster work sampling will be the total cost of the study rather than the variance of the estimator since the total S_i units from a psu chosen at random may be included at little additional cost over observing a single unit in the psu. In other words, the total sample of n instants is not an adequate measure of efficiency since some observations are obtained at less cost than others. This was not the case in the methods of simple random and stratified random work sampling where the cost per observation was approximately the same for both methods. In the case of simple random selection of the m psu's from the total of M psu's, the variance of the estimator from simple random sampling may be applied at the psu level. The estimator when all psu's are equal in size is

$$p = \frac{1}{n} \sum_{i=1}^{m} X_i = \frac{1}{sm} \sum_{i=1}^{m} \sum_{j=1}^{S} X_{ij} , \tag{78}$$

in which S is the uniform cluster size. This estimator is a consistent and unbiased estimator of P (31, Vol. I, p. 245). The actual variance of the estimator is, according to equation (18) in Chapter III,

$$\sigma_p^2 = \left(\frac{1}{S^2}\right)\left(\frac{M-m}{Mm}\right) \sum_{i=1}^{M} \frac{(X_i-\bar{X})^2}{M-1} , \tag{79}$$

where the terms are as defined previously.

The variance, σ_p^2, is not known since it includes unknown population parameters which must be estimated from the sample. The variance estimate is obtained in the same fashion as in previous models by substituting sample data for the unknown universe values, and is

$$s_p^2 = \left(\frac{1}{S^2}\right)\left(\frac{M-m}{Mm}\right) \sum_{i=1}^{m} \frac{(x_i-\bar{x})^2}{m-1} . \tag{80}$$

In view of the fact that the total number of clusters, M, is assumed to be infinite in work sampling, the term $\frac{M-m}{M}$ reduces to unity and the variance estimate becomes

$$s_p^2 = \left(\frac{1}{S^2}\right)\left(\frac{1}{m(m-1)}\right) \sum_{i=1}^{m} (x_i-\bar{x})^2 . \tag{81}$$

<u>Designing a Simple Cluster Work Sample for a Stated Level of Precision</u>

As noted earlier, the cost of additional observations (more than one) within a cluster is usually small in work sampling situations. Hence, the treatment of costs in cluster work sampling which is made later will be a better measure of the efficiency of this method of sampling than the variance of the estimator. However, in order to be able to make confidence interval estimates of the P_i's, the variance of the estimator is still of interest. This variance can also be used for determining the sample size necessary for required levels of precision, the subject of this section. As before, the nature of this requirement will be that the estimate be within 100R per cent of the true value with $1-\alpha$ confidence. Assuming that p is normally distributed, we write

$$RP = K_{\alpha/2}\sigma_p = K_{\alpha/2}\sqrt{\left(\frac{M-m}{M}\right)\left(\frac{1}{S^2 m}\right) \sum_{i=1}^{M} \frac{(X_i-\bar{X})^2}{M-1}} . \tag{82}$$

Since $\frac{M-m}{M}$ is approximately one in work sampling, the above equation can be solved for m, the number of clusters required for the stated precision. Hence,

$$m = \frac{K^2_{\alpha/2} \sum_{i=1}^{M} (X_i-\bar{X})^2}{S^2R^2P^2(M-1)} . \tag{83}$$

As in previous models, this equation cannot be evaluated as it appears above since it includes unknown parameters from the population. However, either historical data or a preliminary sample of m' clusters will provide an estimate of the term $\sum_{i=1}^{M} \frac{(X_i-\bar{X})^2}{M-1}$ by using $\sum_{i=1}^{m'} \frac{(x_i-\bar{x})^2}{m'-1}$, and an estimate of P by

$$p' = \frac{\sum_{i=1}^{m'} \sum_{j=1}^{S_i} x_{ij}}{Sm'} .$$

The approximate number of clusters to include in the sample is then,

$$m \doteq \frac{K^2_{\alpha/2} \sum_{i=1}^{m'} (x_i-\bar{x})^2}{(SRp')^2(m'-1)} . \qquad (84)$$

It should be noted that

$$\frac{\sum_{i=1}^{M} (X_i-\bar{X})^2}{S^2(M-1)} = \sum_{i=1}^{M} \frac{(P_i-P)^2}{M-1} = \sigma^2_{P_i} \qquad (85)$$

is the between cluster variance of the P_i's and is the only contribution to the sampling error of the estimator, p, since clusters chosen in the sample are completely enumerated. This between cluster variance measures the variability in the proportion, P_i, from cluster to cluster of the activity, and also serves as a measure of the interdependence among the members of the clusters, the influence of which is ignored if independence of the within cluster observations is assumed.

When clusters vary in size, such as a team wherein some members spend part of their time doing work not related to the team, the variance of the estimators from the above sampling scheme is considerably more complicated, but can be evaluated. In this case, the estimator is

$$p = \frac{\sum_{i=1}^{m} \sum_{j=1}^{S_i} x_{ij}}{\sum_{i=1}^{m} S_i} = \frac{m\bar{\bar{x}}}{m\bar{\bar{S}}} = \frac{\bar{\bar{x}}}{\bar{\bar{S}}} , \tag{86}$$

where $\bar{S}$ is the average size of the m clusters chosen in the sample and $\bar{x}$ is the average number of subjects per cluster engaged in activity X.

The denominator of this estimator is a random variable as well as the numerator, and the estimator is thus the ratio of two random variables. Since X_i is the number of subjects in the ith cluster engaged in element X and S_i is the number of subjects in the ith cluster, if a random sample of clusters is taken, equation (86) yields an estimate of P_X. The variance of the estimator, p, in equation (86) is approximately

$$\sigma_p^2 \doteq P^2 \left(\frac{M-m}{M}\right)\left(\frac{V_X^2 + V_S^2 - 2\rho V_X V_S}{m}\right), \tag{87}$$

in which

$$V_X^2 = \frac{S_X^2}{\bar{X}^2} = \frac{\sum_{i=1}^{M} (X_i - \bar{X})^2}{(M-1)\ \bar{X}^2}$$

$$V_S^2 = \frac{S_S^2}{\bar{S}^2} = \frac{\sum_{i=1}^{M} (S_i - \bar{S})^2}{(M-1)\bar{S}^2}$$

$$\rho = \sum_{i=1}^{M} \frac{(X_i - \bar{X})(S_i - \bar{S})}{(M-1)\ S_X S_S} \ .$$

The estimator is not unbiased, but according to Hansen, Hurwitz, and Madow (31, Vol. I, p. 163, p. 253), the bias is negligible for moderate sample sizes. Due to the magnitude of sample sizes in work sampling, the bias presents no problem and can safely be ignored. In addition, the approximation of the variance is applicable for all distributions of X_i and/or S_i, and is considered to be good if the sample size is large enough that (31, Vol. I, p. 164)

$$V_{\bar{S}}^2 = \left(\frac{M-m}{M}\right) \frac{V_S^2}{m} \leq 0.0025 \ .$$

Since $\frac{M-m}{M}$ is approximately one in work sampling, this restriction on the variability of $\bar{S}$ amounts to saying that

$$V_S^2 = \frac{S_S^2}{\bar{S}^2} \leq 0.0025\ m, \text{ or } \ S_S \leq 0.05\ \bar{S} \sqrt{m} \ . \qquad (88)$$

For work sampling studies which generally require a relatively large number of observations on the groups, and for small groups of even 4 or 5, this restriction is not critical. For example, if $m = 100$ and

$\bar{S} = 5$, the requirement is that the group size not vary more than about zero to twelve, a variation not likely to be witnessed in work sampling situations.

The estimate of σ_p^2 for the ratio estimator is obtained from the sample by substituting sample calculations for population values in equation (87) as follows (31, Vol. I, p. 176):

$$s_p^2 = \left(\frac{M-m}{M}\right) p^2 \, \frac{v_x^2 + v_S^2 - 2\rho' v_x v_S}{m} , \tag{89}$$

in which p is obtained by equation (73),

$$v_x^2 = \frac{\sum_{i=1}^{m} (x_i - \bar{x})^2}{(m-1)\, \bar{x}^2}$$

$$v_S^2 = \frac{\sum_{i=1}^{m} (S_i - \bar{S})^2}{(m-1)\, \bar{S}^2} , \quad \text{and}$$

$$\rho' = \frac{\sum_{i=1}^{m} (x_i - \bar{x})(S_i - \bar{S})}{(m-1)\, s_x s_S}$$

According to survey sampling theory (31, Vol. I, p. 354), unless the cluster size varies appreciably, the error introduced by using $S = s$ in the variance formulas for constant cluster size is not appreciable. This substitution into equation (78) yields the ratio estimator in equation (86). The approximation is therefore with respect to the variance only. There is good reason to believe that in cluster work sampling, the variation in cluster size will be relatively small. Hence with constant or slightly variable cluster sizes, equations (78) and (79) will be applicable. In case the cluster sizes vary appreciably, the ratio estimator should be used. The substitution of average cluster size for constant cluster size in equations (79) to (84) will yield the approximate method of analysis for the variance of p and the number of clusters to include in the sample for a desired level of accuracy in the results. This case will be used in the remaining sections of this chapter due to its expected prevalence in actual work sampling practice.

<u>A Variance Comparison of Simple Cluster Work Sampling and Simple Random Work Sampling</u>

As already indicated, the total number of elementary units included in the sample for a stated level of precision is usually greater in cluster work sampling than in simple random work sampling. To be specific, if the elements within clusters tend to be alike (positively correlated), the total sample of elementary units is larger for cluster work samples. The magnitude of the difference is greatest when the elements are all alike and becomes zero when the elements are uncorrelated. Should the correlation be negative, cluster work samples will be smaller than simple random work samples for a common level of precision. This phenomenon is illustrated later in this section.

For the case presented in the remaining sections of this chapter -- all psu's equal (or nearly equal) in size, psu's randomly sampled, and 100 per cent enumeration of sampled psu's -- it is expected that the units within a cluster will be positively correlated. In this case, a larger number of elementary units must be collected for a stated precision using cluster sampling as opposed to simple random sampling. However, the number of clusters is usually less than the number of elementary observations in simple random sampling, and the nature of sampling costs within clusters often makes cluster sampling the more efficient of the two procedures when total costs serve as the criterion for comparison. A comparison in terms of costs will appear in the section in which costs are investigated.

The variance of the cluster estimator as given in equation (79) is

$$\sigma_p^2 = \left(\frac{M-m}{Mm}\right) \sum_{i=1}^{M} \frac{(P_i-P)^2}{M-1} ,$$

and since M is infinitely large in work sampling, this reduces to

$$\sigma_p^2 \text{ (cluster)} = \frac{1}{m} \sum_{i=1}^{M} \frac{(P_i-P)^2}{M-1} . \tag{90}$$

When a simple random sample of mS observations is taken (this is the total sample size for the cluster sample of m psu's), the variance of the simple random estimator is given by equation (19), Chapter III, as

$$\sigma_p^2 \text{ (simple random)} = \frac{PQ}{mS} , \tag{91}$$

and the ratio of the variances of simple cluster to simple random sampling is

$$\frac{\sigma_p^2 \text{ (cluster)}}{\sigma_p^2 \text{ (simple random)}} = \frac{\frac{1}{m} \sum_{i=1}^{M} \frac{(P_i-P)^2}{(M-1)}}{\frac{PQ}{mS}} = \frac{\sigma_{p_i}^2}{m\sigma_p^2 \text{ (simple random)}} , \tag{92}$$

which shows that the larger the between cluster variance, the less efficient is cluster sampling when considered on the basis of total observations.

For comparative purposes, however, it is possible to state the variance of the cluster estimator in terms of the variance of the simple random estimator and a measure of the correlation of units within clusters. A slight alteration of an expression given by Cochran (15, p. 203) yields

$$\sigma_p^2 \text{ (cluster)} = \sigma_p^2 \text{ (simple random) } \left\{1 + (S-1)\rho\right\}, \qquad (93)$$

where ρ is the correlation coefficient among units within clusters. Note that when $\rho = 0$, the variance of a cluster estimator from a sample of m clusters of S elements each is equivalent to the variance of a simple random estimator from a sample of mS units. On the other hand, if $\rho = 1$, the cluster variance is S times the simple random variance. And, as pointed out earlier, any time ρ is greater than zero, the cluster sample requires more elementary units than does a simple random sample for a stated precision; i.e., the variance for the cluster estimator is greater than the variance for the simple random estimator for the same total sample size. Also, when ρ is less than zero, the cluster sample will be smaller than the simple random sample for a stated precision. It should be noted that the magnitude of the relative precision is also directly affected by the size of the cluster, S. Since total sample size is a poor measure of relative efficiency between cluster and simple random work sampling, a more detailed discussion of this topic will be deferred to the section on relative costs.

An Illustration of Simple Cluster Work Sampling

The procedure for sampling a population in clusters will be illustrated using Appendix A and comparisons will be made with simple random sampling of the same population. The desired precision of the estimates will be as in the previous illustrations where the objective was to get estimates of the P_i's which were within 10 per cent of the true values with a confidence of 95 per cent. The natural cluster which is obvious in Appendix A is the group of four instants, one for each worker, at each point in time. Hence, $S = 4$ and is constant for all clusters. Cluster minutes will be chosen using random numbers and a random instant will be observed within the chosen minute in keeping with previous practice.

Equation (83) can be used for determining the number of clusters to observe in order that p_i will be within 10 per cent of P_i with 95 per cent confidence. The values of the unknown terms in equation (83) are known for the activity in Appendix A and are

$$\sum_{i=1}^{M} \frac{(X_i-\bar{X})^2}{M-1} = 1.114 \qquad P_1 = 0.1650$$

$$\sum_{i=1}^{M} \frac{(Y_i-\bar{Y})^2}{M-1} = 2.125 \qquad P_2 = 0.2570$$

$$\sum_{i=1}^{M} \frac{(Z_i-\bar{Z})^2}{M-1} = 2.456 \qquad P_3 = 0.5780 .$$

Using equation (83), the number of necessary clusters for each element is

$$m_1 = 974$$

$$m_2 = 840$$

$$m_3 = 177 .$$

Hence, in estimating P_1, it will be possible to obtain the same precision from 974 clusters (completely enumerated) that was obtained from 1944 simple random observations. The 974 clusters contain $4 \times 974 = 3896$ individual observations, approximately twice the number in the simple random sample. A comparison on the basis of costs will be made in the next section as the choice between cluster and simple random sampling schemes must be made on the basis of total costs rather than on the basis of total sample size.

The procedure for carrying out the simple cluster sampling plan is similar to that used in the preceding models. If one chooses $m = 974$ for the total number of clusters to include in the sample, the designation of those points in time at which all S of the subjects will be observed will be made using a table of random numbers. Random minutes will be chosen as before, and the actual observation will be made at an instant naturally randomized within each minute. The entire population of U minutes is subjected to sampling each time and when two or more observations fall in the same minute, further randomization is made using the random number approach. At each designated point in time, all subjects in the group are observed and their individual statuses are noted. The data recorded is simply the number of subjects engaged in each state of the activity at each observation, i.e., X_i, Y_i, Z_i, etc. Equations (78) and (81) are used for calculating the estimate and its approximate variance for each element of the activity. With these values for each element of the activity, confidence interval estimates may be made on the true P_i's in the same manner used in the previous two models. That is,

$$P(p_i - k_{\alpha/2}\, \sigma_{p_i} \leq P_i \leq p_i + k_{\alpha/2}\, \sigma_{p_i}) = 1 - \alpha .$$

In order to better illustrate the sampling distributions of the estimators in cluster work sampling, 200 independent cluster samples of 974 clusters each were drawn from the population in Appendix A. The distributions for p_1, p_2, and p_3 are given in Figure 7. Comparisons of the true parameters in the population and the estimated values are also shown. These results fall well within the scope predicted by the expressions given earlier for the estimators and their variances.

As in the previous cases, when more than one characteristic (element) is estimated from the sample, the foregoing analysis is made for each characteristic and a compromise is reached among the conflicting values of sample size necessary. In the present case, P_1 was chosen as the most important parameter and was estimated with 10 per cent accuracy and 95 per cent confidence. Instead of $R = 0.10$ for the estimators of P_2 and P_3, these are $R_2 = 0.089$ and $R = 0.043$, with the change being due to the sample of 974 clusters.

Cost Factors in Simple Cluster Work Sampling

In the two sampling methods presented previously, simple random and stratified random work sampling, the component costs were essentially the same. The model which gave the desired precision for the

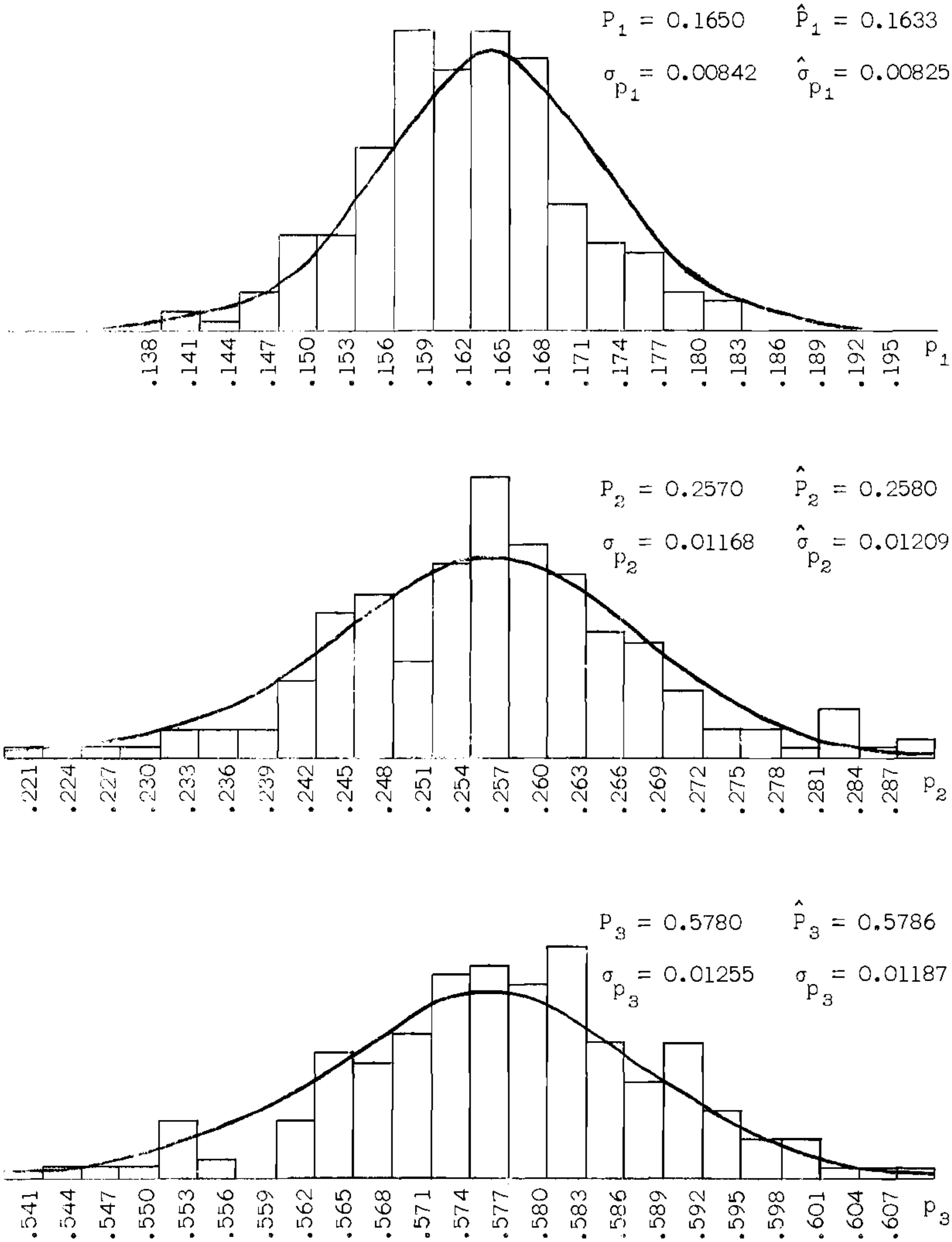


Figure 7. Frequency Distributions of p_1, p_2, and p_3 for 200 Studies Compared with Normal

lowest total cost was thus preferred. Under normal conditions in work sampling, it was observed that stratified sampling was usually preferred to simple random sampling. In the case where the cost per observation is not constant, selection criteria which are based on sample size are not adequate for selecting a plan. Where the cost per additional observation is much less than the cost of a single observation, one can afford to take a larger total sample than where the cost per observation is constant. The overall objective in this case is to obtain maximum precision of results per unit of cost. Hence, the main purpose of cluster work sampling is not to get the most reliable sample in terms of total sample size (elementary units), but to get the most reliable results per unit of cost.

The general nature of costs in cluster work sampling is the same as that for the other models in that there are (1) fixed costs, (2) sampling costs, and (3) "loss due to error" costs, including the usual data processing costs. The latter component of costs, data processing, may be a bit higher in the case of cluster work sampling due to the necessity of analyzing the data in terms of individual clusters rather than in terms of elementary units. This additional cost is not considered to be a significant portion of total costs and is thus not considered in this analysis. The major difference in the costs for cluster work sampling and costs for simple random and stratified random work sampling is in the sampling phase. Since cluster sampling is applicable in those cases where multiple observations may be made simultaneously, the sampling costs may be studied at two stages -- the sampling of psu's and the within psu sampling.

The cost of randomly choosing a single psu and making a single observation will be labeled C_2, and the cost of making more than a single observation in each psu will be C_2' per additional unit. Note that C_2 represents the cost of selecting a psu, observing the activity at the chosen time, and making a single determination of the state of the activity. Hence, C_2 is identical in the cost considerations of all three models presented thus far. C_2' is the cost of making an additional observation at the same time the initial observation is made, hence this cost will be $(S-1)C_2'$ for each psu in the sample since the psu is completely enumerated. The nature of the cost C_2' is of utmost importance in the design of a cluster work sampling study. Because of the nature of work sampling populations, there is reason to believe that the cost C_2' will seldom, if ever, be more than a small fraction of C_2 since the only effort needed for additional observations is simply the recording of the state of another member of the cluster. The major contribution to sampling costs will be reflected in choosing the sample, preparing for data collection, and putting an observer on the scene at the proper instant, which are reflected in C_2. A relationship between C_2 and C_2' will be established as follows:

$$C_2' = f'C_2 \tag{94}$$

in which f' is a fraction between 0 and 1. This function will be used in developing the total cost equation for this model.

Designing a Simple Cluster Work Sample to Minimize Costs or to Achieve a Stated Total Cost

The total cost equation for cluster work sampling may be formulated in terms of the number of primary units chosen in the sample and the relevant cost factors as follows:

$$C = C_1 + m\left[C_2 + (S-1)C_2'\right] + C_e \tag{95}$$

where C, C_1, and C_e are as defined in Chapters III and IV. Total costs pertinent to sampling are (leaving out fixed costs)

$$C' = m\left\{\left[1 + (S-1)f'\right]\right\} C_2 + C_e \tag{96}$$

The expected value of C_e, $E\left[l(e)\right]$, is determined in the same manner as previously. Since e has the same variance as the estimator p, when the loss function $l(e) = ce^2$ is assumed,

$$C_e = c\sigma_e^2 = c\left\{\left(\frac{M-m}{M}\right) \frac{1}{S^2 m} \sum_{i=1}^{M} \frac{(X_i-\bar{X})^2}{M-1}\right\}. \tag{97}$$

Hence, assuming an infinite population of clusters,

$$C' = m\left\{\left[1 + (S-1)f'\right] C_2\right\} + c\left\{\frac{1}{S^2 m} \sum_{i=1}^{M} \frac{(X_i-\bar{X})^2}{M-1}\right\}. \tag{98}$$

To minimize C', we differentiate it with respect to m, equate the result to zero, and solve for m. Thus,

$$\frac{\partial C'}{\partial m} = 0 = \left[1 + (S-1)f'\right]C_2 - \frac{c}{S^2 m^2} \sum_{i=1}^{M} \frac{(X_i-\bar{X})^2}{M-1},$$

and

$$m = \sqrt{\frac{c \sum_{i=1}^{M} \frac{(X_i - \bar{X})^2}{M-1}}{S^2 C_2 \left[1 + (S-1)f'\right]}} \; , \tag{99}$$

which is the number of psu's to include in the sample in order to minimize total costs when the psu's are all equal in size and complete enumeration is carried out. The terms in this expression involving population values will have to be estimated prior to the design of the study using methods previously indicated.

For determining the number of clusters to include in the sample when a stated total cost is specified, the total cost equation is solved for m as follows:

$$C = C_1 + m\left[C_2\left\{1 + (S-1)f'\right\}\right] + \frac{c}{S^2 m} \sum_{i=1}^{M} \frac{(X_i - \bar{X})^2}{M-1} \; , \tag{100}$$

and,

$$m = \frac{(C - C_1) \pm \sqrt{(C - C_1)^2 - \frac{4C_2 c}{S^2}\left\{1 + (S-1)f'\right\} \sum_{i=1}^{M} \frac{(X_i - \bar{X})^2}{M-1}}}{2C_2\left\{1 + (S-1)f'\right\}}$$

Since one is interested in both costs and precision, the necessity for compromise is evident. The values of m calculated for a fixed total cost or for minimum costs may be used in equation (83) to determine the precision which would result from a sample of that size. Likewise, the values of m calculated with equation (83) may be used in the total cost equation above to determine total costs for a stated precision. The nature of the compromise necessary between these two factors is illustrated in the following section.

Choosing a Simple Cluster Sample Based on Both Precision and Costs

Equation (99) may be used to determine the number of clusters to include in the sample in order to minimize costs. For the illustration given previously, some assumed costs will be stated for the purpose of illustrating the influence of costs in cluster work sampling. The ratio of $c/C_s = 2 \times 10^7$ will be assumed as in the previous models ($C_s \equiv C_2$). An additional factor which must be determined is f', the fraction of the cost of an initial observation which applies to additional observations. In work sampling studies, f' is likely to be near zero since little additional effort is required to note the status of all the members of the group as opposed to noting the status of a single member. However, for the purposes of this illustration, a value of $f' = 0.1$ will be assumed in order to adequately allow for whatever additional cost may be incurred. Using these cost values, the following determinations are made using equation (99):

$$m_1 = 1035$$
$$m_2 = 1430$$
$$m_3 = 1535 .$$

The values necessary for the stated accuracy of ± 10 per cent with 95 per cent assurance were shown earlier to be

$$m_1 = 974$$
$$m_2 = 840$$
$$m_3 = 177 .$$

As before, the final determination of how many clusters to include in the sample will be a judgment decision which must be a compromise among the above sets of optimum sample sizes. Looking at the number of clusters for the estimator p_1, it is seen that a sample of 974 clusters will yield an estimate which is within 10 per cent of P_1 with 95 per cent confidence, yet a sample of 1035 clusters will minimize costs. Each practical situation will have its own unique cost and variance characteristics, as was assumed for the example in Appendix A, and will have to be evaluated in the light of these. The final decision on how much sampling will be done must reflect the pertinence of each of these factors and can only be made by those for whom the study is being conducted.

A Cost Comparison of Simple Cluster Work Sampling and Simple Random Work Sampling

A comparison of simple cluster and simple random work sampling, made on the basis of total costs, will indicate the major advantages of sampling clusters of activity. These results cannot be completely generalized; however even under the assumed cost factors of the foregoing illustration, one can observe the relative merits of the two methods of sampling. The three components of cost which make up the total cost equation will be investigated separately. There is little reason for fixed costs to be different for the two plans since virtually the same planning and data processing operations must be carried out regardless of the sampling scheme used. Sampling costs for the two plans which were designed to yield $\pm$ 10 per cent accuracy with 95 per cent assurance are (in terms of C_2), $1944C_2$ for the simple random sample and

$974(1 + 3f')C_2 = 1266C_2$ for the cluster sample. Sampling costs are therefore 53.6 per cent higher for the simple random sample than for the simple cluster sample in this case. It is the cost savings at this phase of the study which makes cluster sampling more efficient than simple random sampling. The final cost component, the "loss due to error," is the same in both cases since the two samples were chosen to yield estimates with the same level of precision. Overall then, the only major cost differences are those which pertain to the selection of the sample.

Although the sampling costs in the foregoing example apply only to the sampling of the activity in Appendix A, and consequently reflect the specific characteristics of that activity, it should be noted that this hypothetical activity is constructed in such a way that the correlation among workers within clusters is positive and relatively large. The value of ρ for this activity is 0.34. The larger this coefficient of correlation, the smaller will be the difference in the sampling costs for the two plans, and vice versa. This follows from the fact that when $\rho = 0$, the variance of the cluster estimator is equal to the variance of the simple random estimator and cluster sampling is more efficient costwise since $(S-1)m$ of the observations in the cluster sample are made at a fraction, f', of the cost of simple random observations. Hence, the illustrative activity in Appendix A is not constructed such that it favors cluster sampling from a cost standpoint. It was observed earlier that due to the high correlation within clusters, approximately twice as many cluster observations had to be made as compared with simple random observations to obtain a stated level of reliability. It was also noted that the correlation within clusters is expected to be positive in most work sampling situations; however, unless the workers tend to perform in unison, the coefficient will not approach its maximum value of one.

In addition, the value of f' assumed above directly affects the relative sampling costs of the two procedures. The value of 0.1 is considered to be very liberal, although the value of f' which would make the sampling costs the same under the two plans is 0.33. In reality, it is expected that f' will be more on the order of 0 to 0.02 in which case simple cluster sampling costs would be even less than those indicated above. Costwise, cluster sampling, where appropriate, is almost certain to be a more economical method for work sampling activities.

Stratified Cluster Work Sampling

The techniques of stratification presented in Chapter IV are as applicable in cluster sampling as they are in simple random sampling. Since clusters (psu's) are sampled by drawing a simple random sample of clusters, gains from stratification may be made if groups of clusters can be formed which are more homogeneous with respect to the characteristic under study than the population of clusters as a whole. The problem is analogous in almost every respect to the case presented in Chapter IV, the primary difference being that a multilaterally classified activity is being sampled instead of one that is unilaterally classified. If one regards the activity in Appendix A as that of a group of four workers working together as a team such that clusters of size four are formed at each point in time, and if differences in the population can be shown to exist between subperiods of the overall activity, the clusters can be stratified to provide a more efficient sampling scheme. The criteria on which to stratify, the administrative problems encountered, and the gains to be expected are the same as those in Chapter IV except the analysis now applies to psu's rather than to elementary units.

The notation for cluster sampling presented earlier in this chapter will suffice if one additional subscript is added to designate strata. The additional notation for strata will be the same as that used in Chapter IV. In the case of stratified clusters, M_h represents the number of psu's in the hth stratum ($h = 1, 2, \ldots\ldots\ldots, L$) and S_{hi} is the number of elementary units in the ith psu in the hth stratum. The expression X_{hij} is the value of X on the jth elementary unit in the ith psu in stratum h. The sums, averages, proportions, etc., for the population and the sample are formed as in simple cluster sampling with the stratum subscript added.

The analyses made in Chapter IV as to how the total sample should be allocated to the strata also apply in cluster sampling. The findings in that case are still applicable since the costs of sampling are not expected to vary from stratum to stratum in stratified cluster sampling and optimum allocations in terms of variances within the strata will be in conflict when more than one characteristic is being estimated. Therefore, the reasons advanced in Chapter IV for using proportional allocation will dictate the same practice in this case. The model developed in the following analysis is for the complete enumeration of clusters drawn at random within strata where proportionate sampling is used within strata.

Combining the estimates of P from stratified random work sampling and simple cluster work sampling, the consistent and unbiased estimator for P is

$$p = \sum_{h=1}^{L} f_h p_h = \frac{\sum_{h=1}^{L} \sum_{i=1}^{m_h} \sum_{j=1}^{S} X_{hij}}{mS}, \tag{101}$$

which is simply the sum of the observations in the sample for which the subjects were engaged in the state of the activity under study divided by the total number of observations in the sample. The variance of this estimator may be found by applying the procedures used in stratified random and simple cluster work sampling and is as follows:

$$\sigma_p^2 = \sum_{h=1}^{L} f_h^2 \sigma_{p_h}^2 = \sum_{h=1}^{L} \left(\frac{f_h}{S}\right)^2 \left(\frac{M_h - m_h}{M_h m_h}\right) \sum_{i=1}^{M_h} \frac{(X_{hi} - \bar{X}_h)^2}{M_h - 1}, \tag{102}$$

which for infinite populations of clusters is

$$\sigma_p^2 = \sum_{h=1}^{L} \left(\frac{f_h}{S}\right)^2 \frac{1}{m_h} \sum_{i=1}^{M_h} \frac{(X_{hi} - \bar{X}_h)^2}{M_h - 1} . \tag{103}$$

Using equation (81), the estimate for this variance from the sample is

$$s_p^2 = \sum_{h=1}^{L} \left(\frac{f_h}{S}\right)^2 \frac{1}{m_h} \sum_{i=1}^{m_h} \frac{(x_{hi} - \bar{x}_h)^2}{m_h - 1} . \tag{104}$$

The Design of a Stratified Cluster Work Sample

The total number of clusters necessary for obtaining an estimate within a desired precision (with proportional allocation within strata) can be determined by using the variance obtained in the preceeding section. Retaining the assumption of normality for the distribution of the estimator, we may write the following relationship for a requirement that p be within 100R per cent of P with $1-\alpha$ confidence.

$$RP = K_{\alpha/2}\sigma_p = K_{\alpha/2}\sqrt{\sum_{h=1}^{L}\left(\frac{f_h}{S}\right)^2 \frac{1}{m_h} \sum_{i=1}^{M_h} \frac{(X_{hi}-\bar{X}_h)^2}{M_h-1}} \qquad (105)$$

$$\frac{R^2P^2}{K^2_{\alpha/2}} = \sum_{h=1}^{L}\left(\frac{f_h}{S}\right)^2 \frac{1}{m_h} \sum_{i=1}^{M_h} \frac{(X_{hi}-\bar{X}_h)^2}{M_h-1} ,$$

and since $f_h = M_h/M = m_h/m$ (proportional allocation to the strata),

$$\frac{R^2P^2}{K^2_{\alpha/2}} = \sum_{h=1}^{L} \frac{f_h}{mS^2} \sum_{i=1}^{M_h} \frac{(X_{hi}-\bar{X}_h)^2}{M_h-1} \qquad (106)$$

$$m = \frac{K^2_{\alpha/2} \sum_{h=1}^{L} \frac{f_h}{S^2} \sum_{i=1}^{M_h} \frac{(X_{hi}-\bar{X}_h)^2}{M_h-1}}{R^2P^2} .$$

It will be necessary to obtain a preliminary sample of m' proportionally stratified clusters from which to estimate the unknown parameters in this equation, unless historical data are available from which these estimates can be made. The procedure for doing this is the same as set forth in the earlier discussion of simple cluster work sampling.

Gains from Cluster Stratification

The gains over simple random sampling by using stratified random sampling or simple cluster sampling have been shown in Chapter IV and the early sections of the present chapter. The former gain is primarily one of precision, the latter is primarily one of cost. When both these features (stratification and clustering) are combined into a single model, gains in both costs and precision may be experienced. Comparing stratified cluster sampling with simple random sampling, a significant gain in sampling costs is realized since the population is multilateral in structure (giving rise to clustering) and the precision of the estimator is improved since clusters are stratified. The comparison made in this section is between simple cluster work sampling and stratified cluster work sampling since the gains of clustering over simple random sampling have been shown earlier. The gain from stratifying clusters is of the same nature as that between stratified and simple random sampling except the analysis is at the psu level.

A comparison of the variances for simple cluster work sampling estimators and stratified cluster work sampling estimators can be made in terms of analysis of variance concepts in the same manner used in Chapter IV. The variance of the population of clusters is divided into two components in the stratification process, the variance between strata and the variance within strata. Only the latter is reflected in a stratified cluster sample subsequently chosen from the population. The variance of the estimator from a sample of m clusters was shown in equation (90) to be

$$\sigma_p^2 \text{ (simple cluster)} = \frac{\sum_{i=1}^{M} (P_i-P)^2}{m(M-1)} = \frac{\sigma_{P_i}^2}{m} . \tag{107}$$

This variance can be written in terms of the stratified population as

$$(M-1)\sigma_{P_i}^2 = \sum_{h=1}^{L} \sum_{i=1}^{M_h} (P_{hi}-P)^2 , \tag{108}$$

the right side of which can further be divided into the between strata and the within strata sums of squares, using the same algebraic identity previously quoted. It follows that

$$\begin{aligned}(M-1)\sigma_{P_i}^2 &= \sum_{h=1}^{L} \sum_{i=1}^{M_h} (P_{hi}-P_h)^2 + \sum_{h=1}^{L} M_h(P_h-P)^2 \\ &= \sum_{h=1}^{L} (M_h-1)\sigma_{p_h}^2 + \sum_{h=1}^{L} M_h(P_h-P)^2 .\end{aligned} \tag{109}$$

Due to the infinite populations of clusters involved, $M-1$ and M_h-1 are taken as M and M_h respectively. Division by mM then yields

$$\frac{\sigma_{P_i}^2}{m} = \sum_{h=1}^{L} \frac{f_h\sigma_{p_h}^2}{m} + \sum_{h=1}^{L} \frac{f_h(P_h-P)^2}{m} , \tag{110}$$

which is

$$\sigma_p^2 \text{ (simple cluster)} = \sigma_p^2 \begin{pmatrix}\text{proportionally} \\ \text{stratified} \\ \text{clusters}\end{pmatrix} + \sum_{h=1}^{L} \frac{f_h(P_h-P)^2}{m} . \tag{111}$$

This relationship is analogous to the one obtained in Chapter IV to show the gain of stratified random sampling over simple random sampling. The gain in using stratified cluster sampling as opposed to simple cluster sampling is obviously dependent upon the magnitudes of the differences in the stratum to stratum variances of P, and is maximum when these differences are maximum.

An Illustration of Stratified Cluster Work Sampling

As with previous models, this method of sampling will be illustrated by sampling the activity in Appendix A. For the same reasons stated in Chapter IV, the population will be divided into ten strata, each day representing an individual stratum. The sample will be designed to yield estimates of the P_i's which are within ± 10 per cent of the true value 95 per cent of the time. The stratum variances will be needed in calculating the necessary sample sizes for the three estimates using equation (106) and are given in Table 3. The sample size calculations yield the following values:

$$\begin{aligned} m_1 &= 507 \\ m_2 &= 206 \\ m_3 &= 57 \,. \end{aligned} \tag{112}$$

As compared to the sample sizes required for the same precision when simple cluster sampling is used, i.e.,

$$\begin{aligned} m_1 &= 974 \\ m_2 &= 840 \\ m_3 &= 177 \,, \end{aligned}$$

Table 3. Stratum Variances (Clusters) for Appendix A

Stratum	$\sigma^2_{p_{h1}}$	$\sigma^2_{p_{h2}}$	$\sigma^2_{p_{h3}}$
1	.0186	.0240	.0471
2	.0304	.0359	.0618
3	.0788	.0791	.0304
4	.0382	.0170	.0488
5	.0066	.0524	.0413
6	.0517	.0218	.0661
7	.0437	.0476	.0770
8	.0377	.0318	.0584
9	.0150	.0251	.0106
10	.0386	.0186	.0503

it is obvious that stratification yields a rather significant gain in variance efficiency. The distributions of the estimators p_1, p_2, and p_3 from 200 proportionally stratified cluster samples from the activity in Appendix A are shown in Figure 8. The sample size corresponding to p_1 was chosen, allowing a simple random sample of clusters of size 51 from each stratum. Actual versus estimated values of the parameters and the estimator variances are also given in Figure 8. The results reflect the validity of the foregoing model.

Cost Factors in Stratified Cluster Work Sampling

An analysis of costs in this case would parallel the cost analysis for stratified random sampling presented in Chapter IV. The preliminary

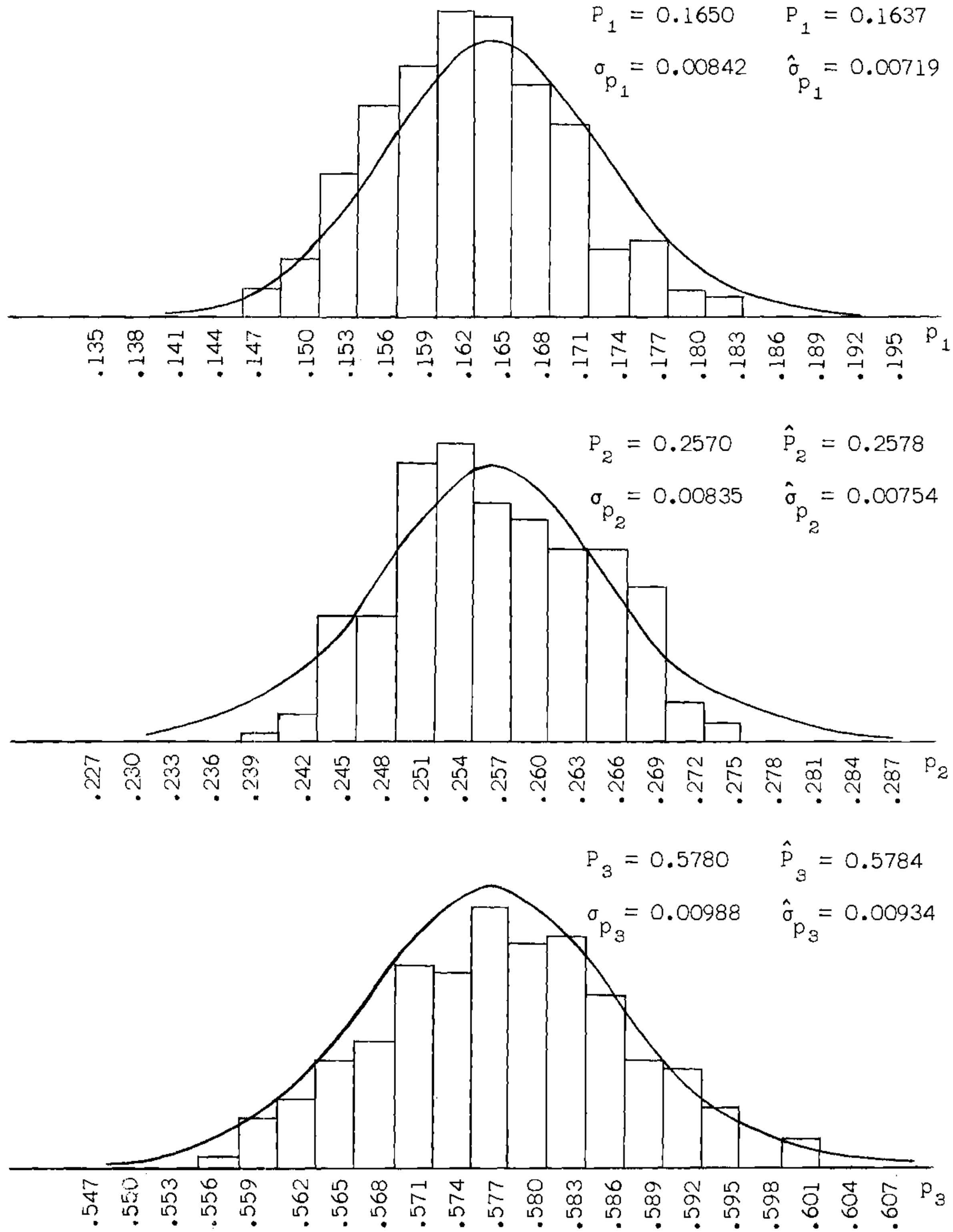


Figure 8. Frequency Distributions of p_1, p_2, and p_3 for 200 Studies Compared with Normal

remarks made there with respect to the nature of these costs are valid in the present case and will not be repeated. The total cost equation in stratified cluster work sampling is

$$C = C_1 + \sum_{h=1}^{L} m_h c_h + C_e \text{ ,} \tag{113}$$

where C_1 and C_e are the fixed costs and the "loss due to error" costs as before. The sampling costs are represented by the sum of the sampling costs within each stratum, where m_h is the number of clusters in the sample from the hth stratum and c_h is the corresponding cost of including a single cluster from the hth stratum. As in Chapter IV, all strata are expected to be equal in terms of sampling costs for work sampling, hence this sum can be replaced by $m\left[1 + (S-1)f'\right] C_2$. The term C_e will assume the loss function used in earlier chapters and therefore has an expected value which is a constant times the variance of the stratified cluster estimator. Introducing these values into the total cost equation yields

$$C = C_1 + mC_2\left[1 + (S-1)f'\right] + c \sum_{h=1}^{L} \frac{f_h}{mS^2} \sum_{i=1}^{M_h} \frac{(X_{hi}-\bar{X}_h)^2}{M_h-1} \text{ .} \tag{114}$$

Choosing m to minimize these total costs yields

$$\frac{\partial C}{\partial m} = 0 = C_2\left[1 + (S-1)f'\right] - \frac{c}{m^2} \sum_{h=1}^{L} \frac{f_h}{S^2} \sum_{i=1}^{M_h} \frac{(X_{hi}-\bar{X}_h)^2}{M_h-1} \text{ ,} \tag{115}$$

and

$$m = \sqrt{\frac{\frac{c}{S^2} \sum_{h=1}^{L} f_h \sum_{i=1}^{M_h} \frac{(X_{hi}-\bar{X}_h)^2}{M_h-1}}{C_2\left[1 + (S-1)f\right]}} \quad .$$

As in previous cases, if the total cost is fixed, equation (114) is solved for m to yield the proper number of clusters to draw into the sample. The values of m arrived at in these cases may be used in equation (106) to determine the precision of the estimator when costs are minimized or when total costs are fixed. Likewise, when the sample is designed to yield a stated precision by using equation (106), the resulting value of m may be substituted in equation (114) to determine the total cost of the study.

The decision as to what the final sample size will be must be made as before by determining which estimators are most important as well as whether the precision of the estimator or overall costs of the study is to be emphasized. In the present case, using the assumed cost values stated earlier, equation (115) yields the following sample sizes necessary to minimize costs:

$$\begin{aligned} m_1 &\doteq 186 \\ m_2 &\doteq 184 \\ m_3 &\doteq 218 \,. \end{aligned} \tag{116}$$

These are compared with the values in equation (112) in arriving at the compromise necessary between costs and precision.

Limitations of Cluster Work Sampling

Cluster work sampling, in general, is a more efficient method for sampling the activities of a group of subjects than is simple random work sampling. This results from the relatively low per unit cost of multiple observations as compared with single observations. The ability to make multiple observations simultaneously without the introduction of error by the analyst as well as by the subjects is a function of the analyst's ability as well as the number of subjects in the cluster. These limitations are obviously less significant with smaller clusters; however the theory is applicable to any size cluster.

The slightly more involved calculations in processing the data of a cluster sample places a greater demand on the analyst. However, after a period of familiarization, these calculations should come as natural as the currently used simple random calculations.

CHAPTER VI

MULTI-STAGE WORK SAMPLING

The Scope of this Chapter

The work sampling models developed in Chapters III through V pertain to the sampling of activities which are normally located within a fairly narrow geographical confine. Those developments are readily adaptable to manual data processing and control. The introduction of more refined techniques such as those considered in the allocation of a stratified sample, for example, obviously adds to the computational complexity of a study. As pointed out in the section dealing with allocation, the statistical gains are often not worth the added complexity in the analysis. This is not to indicate that improvements in work sampling procedures which introduce more complex analyses should be arbitrarily abandoned because of this. Instead, new approaches should be evaluated in terms of whether or not they significantly increase the value of the data or significantly reduce the costs of the study. Indeed there are situations where complex analyses cannot be avoided; multi-stage work sampling is such a case. There are numerous refinements in sampling techniques available in the literature which may become more suitable to work sampling as electronic data processing becomes an integral part of a system and the practitioners' level of statistical knowledge is improved. In keeping with the fourth specific objective listed in Chapter I, this chapter treats a refinement in work sampling which is expected to become more pertinent as systems analyses become more complex.

Electronic data processing has already become an important tool for many large organizations and is becoming more readily accessible to smaller organizations through electronic data processing service centers. The systems concept (or operations research approach) has become an important aspect of top level managerial planning and control. It has thus placed a unique demand on the management scientist, including the industrial engineer, for data pertinent to large complex systems of men, materials, and equipment. Work sampling has been a useful tool for the collection of such data in part, but in its traditional form (simple random sampling and binomial theory) it becomes cumbersome and expensive as the scope of the problem enlarges. The electronic computer has assisted in the analysis of systems models, including work sampling data (36), and has repeatedly proven its worth when properly utilized. This chapter presents the development of a work sampling model in which sampling is performed at several stages of the population. The computational analyses of the data obtained in such a fashion can best be handled by computers and the method would not generally be employed where such facilities are not available.

The collection of data pertinent to total systems has become more important today than in the past for purposes of budgeting, planning, staffing, control, and many other broad managerial functions. Thus, tools for obtaining and evaluating such data are of special interest. Multi-stage sampling methods in work sampling offer the industrial engineer an effective, economical, and statistically valid tool which can be applied to the complex problems of data collection and analysis at the broader systems level when computing facilities are at his disposal. This method of sampling has been used successfully in other fields of endeavor (15, 20, 31) and has proven to be very effective. Its major characteristic is that instead of taking a single sample of the ultimate sampling units in a population, one can do equally as well in terms of estimators with calculable precision, and considerably better economically, to "nest" the population in several stages and sample each stage successively until the ultimate unit of interest is sampled. The sampling can be carried out in various ways, depending primarily on whether or not estimates of the parameters under study are desired at each stage or just for the overall study. For example, a large manufacturer with multiple factories might be interested in knowing the proportion of total supervisory time which is spent in the state "instruct subordinates" in the total organization. An estimate of this factor (as well as others) could be obtained by sampling, in turn, factories, divisions within factories, departments within divisions, offices within departments, individuals within offices, time periods of the individuals' activities, and instants within these periods. This chapter will give the pertinent procedure for developing work sampling models to be used in such broad studies and will discuss the cost analyses necessary for determining the optimal amounts of sampling at each stage where possible.

There is considerable flexibility in selecting a method of sampling at each stage (usually one of the procedures presented in the preceding chapters); however in this case it is not possible to develop a single general estimator and its variance. The method of sampling in stages also becomes more complex if the units at a given stage are not equal in size. The expressions for such estimators are functions of the sampling schemes chosen at the various levels of sampling. For example, if there are four levels of sampling and four sampling schemes to choose from at each level, then there are two hundred and fifty-six unique estimators possible, depending on the sampling scheme used at each level.

The complexity of formulating the estimator and its variance for the overall model is directly related to the complexity of the components of variance contributed by the sampling schemes used at the various levels of sampling. Because of this fact, simple random sampling will be preferred at a given stage unless significant gains are obviously possible when a more complex form of sampling is used. The approach in this chapter will be to present a general model equation somewhat as in the analysis of variance, to write the expression for the general estimator and its variance when simple random sampling is used at all stages, and to show how deviations from this general scheme affect the resulting estimators and their properties.

Multi-Stage Work Sampling

The observations on an activity when r stages of sampling are used may be written as follows:

$$X_{abc\ldots r} = \mu + \alpha_a + \beta_{(a)b} + \cdots\cdots\cdots + \gamma_{(a\ldots q)r} , \qquad (117)$$

in which μ represents the general average common to all observations and each successive term in this general equation represents the contribution of the stage of sampling represented by the last letter in the subscript. Note that the subscript notation is that for nested factors in an analysis of variance since each successive stage is completely nested in the previous stages. Nested in this sense simply means that a given subscript is represented in each realization of all successive subscripts.

In its simplest form, multi-stage work sampling can be depicted by the method of choosing a sampling unit discussed in Chapter III. The method of choosing random minutes (with replacement) and then random instants within the minutes chosen amounted to a simple random selection of instants. In essence, the procedure was carried out by sampling at two stages, minutes and instants. It would have been possible to choose minutes at random without replacement and instants within minutes and then to express the variance of the estimator, p, as a sum of two components of variance -- the between minutes component and the within minutes component. The model equation in that case would have been

$$X_{ab} - \mu = \alpha_a + \beta_{(a)b} ,$$

in which α_a represents a variation due to minutes and $\beta_{(a)b}$ represents a variation due to instants within minutes. Obviously, the result would not have been simple random work sampling (which was the model presented in Chapter III) and the added complexity of sampling in stages would have precluded its use in that case. Also, in cluster work sampling (Chapter V), if clusters are not completely enumerated, further sampling from the clusters drawn into the sample would result in a two stage sampling scheme. This approach might be preferred if it were not desirable to record the status of each subject in a cluster due to the similarity of the observations. The idea of sampling in stages is clearly evident from these examples. In general, the reasons for using multi-stage sampling are synonymous with the reasons advanced in discussing the choice of a sampling unit in Chapter III -- administrative convenience and reduced costs in administering the study.

With the introduction of a multi-stage work sampling model, it becomes more evident why work sampling was cast in terms of finite population sampling in the earlier portions of this study. All but the final stage of sampling in multi-stage work sampling will be from finite populations and due to the limited sizes of the populations expected at these levels, the correction factors for this method of sampling will generally have to be included in the variance calculations. The final stage will involve the sampling of instants of which there is always assumed to be an infinite number. These concepts will be more obvious as the construction of a sampling model progresses.

As indicated earlier, the use of multi-stage work sampling is likely to be applicable only at the broader levels of data collection and analysis. The increasing emphasis on this type of study, however, is evident. Also, the phases of activity which are of interest will be broad since they must be common to all segments of the population. Such phases of activity as idle time, telephone usage, conferences with subordinates, etc., would be typical. The measurement of the proportions of time spent in such states of activity through the employment of multi-stage sampling techniques requires that the total population be divided into a number of mutually exclusive first stage units, that each of these units be further divided into a number of mutually exclusive second stage units, these in turn must be divided into a number of third stage units, etc., for as many levels or stages of the activity as the given situation dictates. The sampling of such a population is simplest when units at a given stage are all equal in size and are sampled in a simple random manner with equal size subsamples from each unit. While a work sampling population is seldom expected to meet this requirement in its natural state, it is conceivable that the units may sometimes be re-defined to approximate this situation. In the development which follows, this case will be used as a focal point for introducing more realistic population structures and sampling schemes. This approach will afford a means for observing the nature of the complexities introduced by whatever variations from the general model may be encountered.

Estimators for the General Multi-Stage Model

The proportions of total time devoted to well defined elements of an activity continue to be the parameters of interest. The estimates of these parameters (p_i is an estimate of P_i, the proportion of time spent performing the ith element) are functions of samples of observations made on the activity at random points in time. As in the previous models, the difference being introduced is in the sampling plan. A three stage sample will be used later for illustrating the costs involved in a specific multi-stage work sampling application; however, expressions of the estimators and their variances will first be given for a general r stage population. The actual number of stages possible in a study is unlimited but would be expected to be only a few in most work sampling situations. The first stage or level of the population in an r stage sampling plan would consist of N_1 units numbered

$a = 1, 2, \ldots, N_1$, at the second stage each of these N_1 units would consist of N_2 units numbered $b = 1, 2, \ldots, N_2$, etc., and at the rth stage, each of the (r-1)st stage units would consist of N_r units numbered $r = 1, 2, \ldots, N_r$. The value of a characteristic at, say, the rth instant of time for the (r-1)st worker's activity, ..., in the bth department of the ath factory would be designated $X_{abc\ldots r}$ in keeping with the general model. This would be a zero-one variable, its value depending on whether or not the Xth state of the activity was observed. Other states of the activity would be denoted similarly as $Y_{abc\ldots r}$, etc. The population of units at the final stage of the activity can be assumed to consist of an infinite number of instants $(N_r = \infty)$ while the population size at all other stages would be finite. Finite notation for sampling at the last stage will be used in the development of the estimators, as in the earlier models, and the limit as the size of the population at this stage approaches infinity will subsequently be used.

The proportion of the total population of time spent in state X would be

$$P = \frac{\sum_{a=1}^{N_1} \sum_{b=1}^{N_2} \cdots \sum_{r=1}^{N_r} X_{abc\ldots r}}{N_1 N_2 \ldots N_r} . \tag{118}$$

An estimate of this proportion of time devoted to the Xth state of the activity would be

$$p = \frac{\sum_{a=1}^{n_1} \sum_{b=1}^{n_2} \cdots \sum_{r=1}^{n_r} x_{abc \ldots r}}{n_1 n_2 \cdots n_r}, \tag{119}$$

in which n_k is the sample size at the kth stage of the activity. The product of these sample sizes for all stages of the population is the total number of instantaneous observations made on the activity.

The estimator, p, is an unbiased estimator of the parameter P since simple random sampling is assumed at each stage. If methods other than simple random sampling are used, the observations which make up the estimator must be weighted properly to insure unbiasedness. This practice is illustrated in the cases which follow the general model.

The variance of the estimator is made up of contributions from each stage of sampling, the components being directly additive if the sampling at each stage is independent, a requirement which is easily fulfilled. The variance expression for the estimator given in equation (119) is obtained by an extension of a form given by Cochran (15, p. 230) as:

$$s_p^2 = \frac{N_1 - n_1}{N_1 n_1} s_1^2 + \frac{N_1 N_2 - n_1 n_2}{N_1 N_2 n_1 n_2} s_2^2 + \ldots + \frac{N_1 N_2 \ldots N_r - n_1 n_2 \ldots n_r}{N_1 N_2 \ldots N_r n_1 n_2 \ldots n_r} s_r^2, \tag{120}$$

in which s_k^2 is the variance of the component associated with the units at the kth stage of sampling. These components of variance reflect the assumption of simple random sampling of equal size units at each level. Since the estimator and its variance depend upon the types of sampling used at each level, these cannot be determined in a multi-stage work sampling model until after the choices of sampling schemes have been made at each level of sampling.

The variance expressions, S_k^2, are the variances among the N_k units at the kth stage of sampling. These population variances, which have to be estimated to give an estimate of S_p^2, equation (120), have been defined for a three stage sample by Cochran (15, p. 230) in analysis of variance terminology. An extension to the general case of r stages is as follows:

	d.f.	m.s.
Between 1st stage units	N_1-1	$\dfrac{N_2N_3\ldots N_r\sum_{a=1}^{N_1}(P_a-P)^2}{N_1-1} \equiv \begin{bmatrix} S_r^2+N_rS_{r-1}^2+ \\ \ldots+N_rN_{r-1}\ldots N_4S_3^2+ \\ N_rN_{r-1}\ldots N_4N_3S_2^2+ \\ N_rN_{r-1}\ldots N_4N_3N_2S_1^2 \end{bmatrix}$
Between 2nd stage units	$N_1(N_2-1)$	$\dfrac{N_3N_4\ldots N_r\sum_{a=1}^{N_1}\sum_{b=1}^{N_2}(P_{ab}-P_a)^2}{N_1(N_2-1)} \equiv \begin{bmatrix} S_r^2+N_rS_{r-1}^2+ \\ \ldots+N_rN_{r-1}\ldots N_4S_3^2+ \\ N_rN_{r-1}\ldots N_4N_3S_2^2 \end{bmatrix}$
Between 3rd stage units	$N_1N_2(N_3-1)$	$\dfrac{N_4N_5\ldots N_r\sum_{a=1}^{N_1}\sum_{b=1}^{N_2}\sum_{c=1}^{N_3}(P_{abc}-P_{ab})^2}{N_1N_2(N_3-1)} \equiv \begin{bmatrix} S_r^2+N_rS_{r-1}^2+\ldots \\ +N_rN_{r-1}\ldots N_4S_3^2 \end{bmatrix}$
.	.	
.	.	
.	.	
Between rth stage units	$N_1N_2\ldots N_{r-1}(N_r-1)$	$\dfrac{\sum_{a=1}^{N_1}\sum_{b=1}^{N_2}\sum_{c=1}^{N_3}\ldots\sum_{r=1}^{N_r}(P_{abc\ldots r}-P_{abc\ldots r-1})^2}{N_1N_2N_3\ldots N_{r-1}(N_r-1)} \equiv \left[S_r^2\right]$

Estimates of the mean squares in the above array are easily obtained by substituting sample values for the population parameters. For example,

$$\widehat{MS_1} = \frac{n_2 n_3 \cdots n_r \sum_{a=1}^{n_1} (p_a - p)^2}{n_1 - 1}, \tag{121}$$

is an unbiased estimate of the mean square for between first stage units. The extension to the other mean squares may be accomplished with no difficulty. Hence, an unbiased estimate of the variance of p, the estimator from a multi-stage work sample, can be obtained from this series of mean square estimates, and is:

$$s_p^2 = \hat{S}_p^2 = \frac{1}{n_1 n_2 \cdots n_r}\left[\left(\frac{N_1-n_1}{N_1}\right)\widehat{MS_1} + \left(\frac{N_2-n_2}{N_2}\right)\left(\frac{n_1}{N_1}\right)\widehat{MS_2} + \left(\frac{N_3-n_3}{N_3}\right)\left(\frac{n_1 n_2}{N_1 N_2}\right)\widehat{MS_3} + \cdots + \left(\frac{N_r-n_r}{N_r}\right)\left(\frac{n_1 n_2 n_3 \cdots n_{r-1}}{N_1 N_2 N_3 \cdots N_{r-1}}\right)\widehat{MS_r}\right]. \tag{122}$$

The proof of this statement is obtained by substituting the expected values of the $\widehat{MS_k}$ into equation (122) to yield equation (120).

Equations (118) and (120) represent the true values of P and the variance of the estimator given in equation (119). Equation (122) gives an unbiased estimate of this true variance and provides the means for estimating the fraction P and its variance for the general multi-stage work sampling model. These estimates may be used, as in previous models, for setting confidence interval estimates on the parameters.

A cost analysis similar to those for previous models can be made to determine the optimal amounts of sampling at each stage. This analysis is indicated in the next section, and is restricted to a three stage model in order to make the procedure more explicit; it can be extended in an obvious way.

Multi-Stage Design Considering Either a Cost or a Variance Constraint

The primary objective in designing the sample for multi-stage work sampling is the minimization of costs for stated degrees of precision, or conversely, the maximization of precision for stated levels of costs. The optimum design from an overall cost standpoint without regard to the precision of the estimator would be the set of sample values (one for each stage) which achieved minimum total cost. These three segments of the cost problem are treated here, in the above order.

Cost components remain the same as in previous models and the cost notation introduced in the previous chapters is still sufficient. C_1 represents the fixed costs of the study, C_s is the sampling costs, and C_e represents the cost of "losses due to error" in the estimate. The factor C_1 will not influence the sample design, and C_e will continue to be taken as the expected loss due to error where the loss function is ce^2. The expected value of this loss, it will be recalled, is just $c\sigma_p^2$. The structure of C_s will reflect the cost of sampling at the various levels in the population. Assuming that c_i is the cost of a single observation at the ith level, C_s may be written as follows:

$$C_s = c_1n_1 + c_2n_1n_2 + c_3n_1n_2n_3 \; . \tag{123}$$

The total cost equation for the multi-stage model is

$$C = C_1 + C_s + C_e \tag{124}$$

$$= C_1 + c_1 n_1 + c_2 n_1 n_2 + c_3 n_1 n_2 n_3 + c \left\{ \frac{N_1 - n_1}{N_1 n_1} S_1^2 + \frac{N_1 N_2 - n_1 n_2}{N_1 N_2 n_1 n_2} S_2^2 + \frac{N_1 N_2 N_3 - n_1 n_2 n_3}{N_1 N_2 N_3 n_1 n_2 n_3} S_3^2 \right\} .$$

A well-known procedure is used for optimizing the sample design subject to a stated constraint (15, p. 225ff). For the purposes of this analysis, we let $k_1 = n_1$, $k_2 = n_1 n_2$, and $k_3 = n_1 n_2 n_3$. Then for a stated precision in p, say v, to minimize the total cost of the study, we use the Lagrange multiplier and minimize

$$G = c_1 k_1 + c_2 k_2 + c_3 k_3 + \lambda \left\{ \left(\frac{1}{k_1} - \frac{1}{N_1} \right) S_1^2 + \left(\frac{1}{k_2} - \frac{1}{N_1 N_2} \right) S_2^2 + \left(\frac{1}{k_3} - \frac{1}{N_1 N_2 N_3} \right) S_3^2 \right\} \tag{125}$$

by differentiating with respect to each k_i and solving the resulting equations for n_1, n_2, and n_3. (Note that the terms in the cost equation which do not appear in equation (125) are constant and hence drop out when derivatives are taken.) The resulting equations are:

$$\frac{\partial G}{\partial k_1} = c_1 - \frac{\lambda}{k_1^2} S_1^2 = 0 \tag{126}$$

$$\frac{\partial G}{\partial k_2} = c_2 - \frac{\lambda}{k_2^2} S_2^2 = 0$$

$$\frac{\partial G}{\partial k_3} = c_3 - \frac{\lambda}{k_3^2} S_3^2 = 0 .$$

From these equations we get

$$k_1 = \sqrt{\frac{\lambda}{c_1}}\ S_1 = n_1$$

$$k_2 = \sqrt{\frac{\lambda}{c_2}}\ S_2 = n_1 n_2$$

$$k_3 = \sqrt{\frac{\lambda}{c_3}}\ S_3 = n_1 n_2 n_3$$

and

$$n_2 = \frac{k_2}{n_1} = \frac{\sqrt{c_1}\,S_2}{\sqrt{c_2}\,S_1}$$

$$n_3 = \frac{k_3}{k_2} = \frac{\sqrt{c_2}\,S_3}{\sqrt{c_3}\,S_2} .$$

The value of n_1 is obtained by substituting the above values of n_2 and n_3 into the expression for the variance, which was fixed at V. The result is as follows:

$$V = \left(\frac{1}{n_1} - \frac{1}{N_1}\right) S_1^2 + \left(\frac{1}{n_1 n_2} - \frac{1}{N_1 N_2}\right) S_2^2 + \left(\frac{1}{n_1 n_2 n_3} - \frac{1}{N_1 N_2 N_3}\right) S_3^2 ,$$

$$\frac{S_1^2}{n_1} + \frac{S_2^2}{n_1 n_2} + \frac{S_3^2}{n_1 n_2 n_3} = V + \frac{S_1^2}{N_1} + \frac{S_2^2}{N_1 N_2} + \frac{S_3^2}{N_1 N_2 N_3} = V' ,$$

$$n_1 = \frac{1}{V'} \left\{ S_1^2 + \frac{S_1 S_2 \sqrt{c_2}}{\sqrt{c_1}} + \frac{S_1 S_3 \sqrt{c_3}}{\sqrt{c_1}} \right\} .$$

The values of n_1, n_2, and n_3 can be evaluated using estimates of S_1^2, S_2^2, and S_3^2 provided by equation (121) from a previous set of data or from a preliminary study.

Instead of minimizing the total cost of a study subject to a fixed variance for the estimator, it may be desirable to fix the total cost at, say T, and find the sample sizes which minimize the variance of the estimator subject to this constraint. This may be accomplished in the same manner that was used in the previous problem. The procedure in this case is to minimize

$$G = \lambda \left\{ C_1 + k_1c_1 + k_2c_2 + k_3c_3 + c\left[\left(\frac{1}{k_1} - \frac{1}{N_1}\right)S_1^2 + \left(\frac{1}{k_2} - \frac{1}{N_1N_2}\right)S_2^2 + \left(\frac{1}{k_3} - \frac{1}{N_1N_2N_3}\right)S_3^2\right]\right\} + \left(\frac{1}{k_1} - \frac{1}{N_1}\right)S_1^2 + \left(\frac{1}{k_2} - \frac{1}{N_1N_2}\right)S_2^2 + \left(\frac{1}{k_3} - \frac{1}{N_1N_2N_3}\right)S_3^2 \tag{127}$$

with respect to k_1, k_2, and k_3 and again solve for the values of n_1, n_2, and n_3. It follows that

$$n_2 = \frac{k_2}{n_1} = \frac{S_2\sqrt{c_1}}{S_1\sqrt{c_2}}$$

$$n_3 = \frac{k_3}{k_2} = \frac{S_3\sqrt{c_2}}{S_2\sqrt{c_3}} .$$

The value of n_1 is found by solving the total cost equation, which was set equal to T. It follows that

$$T = C_1 + n_1(c_1 + n_2c_2 + n_2n_3c_3) + \frac{1}{n_1}\left\{c\left(S_1^2 + \frac{S_2^2}{n_2} + \frac{S_3^2}{n_2n_3}\right)\right\} - c\left\{\frac{S_1^2}{N_1} + \frac{S_2^2}{N_1N_2} + \frac{S_3^2}{N_1N_2N_3}\right\}.$$

Accumulating the terms not involving sample sizes and designating this constant as B, i.e.,

$$B = C_1 - T - c\left\{\frac{S_1^2}{N_1} + \frac{S_2^2}{N_1N_2} + \frac{S_3^2}{N_1N_2N_3}\right\},$$

and substituting for n_2 and n_3, we can rewrite the expression as a quadratic function in n_1 as follows:

$$\underbrace{\left(\left\{c_1 + \frac{S_2\sqrt{c_1c_2}}{S_1} + \frac{S_3\sqrt{c_1c_3}}{S_1}\right\}\right)}_{A} n_1^2 + Bn_1 + \underbrace{\left(c\left\{S_1^2 + \frac{S_1S_2\sqrt{c_2}}{\sqrt{c_1}} + \frac{S_1S_3\sqrt{c_3}}{\sqrt{c_1}}\right\}\right)}_{D} = 0$$

from which

$$n_1 = \frac{-B \pm \sqrt{B^2 - 4AD}}{2A}.$$

These expressions for n_1, n_2, and n_3 can be evaluated as indicated previously by estimating the variance, S_k^2, using equation (121). As before, this is usually accomplished by using data from previous studies, although a preliminary study may be conducted for this purpose.

It should be noted that optimum sample sizes for the two cases presented above are the same for all levels of sampling except the first. If this procedure is extended to the general r stage case, the optimum sample size for the kth stage $(k > 1)$ is

$$n_K = \frac{\sqrt{c_{K-1}}\ S_K}{\sqrt{c_K}\ S_{K-1}}, \tag{128}$$

and the value of n_1 is always obtained by solving the cost or variance equation, whichever has been constrained.

If there are no cost or variance constraints on the problem (which is unlikely), the cost of the study is minimized by taking partial derivatives of equation (124) with respect to each n_k and solving the resulting set of equations simultaneously for the optimum sample size at each level.

Deviations from the General Model

The restrictions placed on the foregoing development of a general multi-stage work sampling model, viz., that units are equal in size at any specified level of sampling and that simple random sampling is used at each level, severely limits its application. Nevertheless, this general model provides a focal point around which discussion may be based whenever deviations are made from the general case. If units of approximately the same size are formed at each stage, and if simple random sampling is used exclusively, the general model will be applicable in providing a good approximate solution. In this case, the averages of unit sizes at each stage are used as in Chapter V where average cluster sizes were suggested.

The major deviations from the general model which are expected to occur in actual practice fall into two broad categories: (1) Units at a given stage may vary in size, and (2) Methods other than simple random sampling may be used at some stage(s). The latter deviation is easiest to cope with in terms of showing its influence on the estimator and its characteristics. Adequate theory exists for utilizing methods of sampling other than simple random sampling in a multi-stage population. However, when units vary in size, explicit methods of cost optimization are not available, although numerous estimators and their variances have been developed for use in this case (15, Chapter 11).

The objective of this section will be to show the approach where deviations can be adequately treated, and to indicate the nature of the complexities introduced in other cases. The influence of using methods of sampling other than simple random sampling will be treated first.

Variations in Sampling Scheme

As indicated earlier, it is conceivable that any of the methods of sampling presented in Chapters III through V can be employed at any level of multi-stage work sampling. It is logical that variations in the sampling schemes for work sampling applications are of most importance at the last stage because it is at this stage that the activity itself is sampled. The units at this stage are time oriented as in all previous models. Units at all stages prior to the last are subdivisions of the organization structure of the activity, e.g., factories, work stations, etc. The technique of clustering is not likely to be applicable in work sampling except in cases where groups of subjects are observed at the final stage. Stratification may offer gains at any stage, although in this case, the most significant gains are likely to come from stratifying the grossest sampling units (first stage) and/or the time units (last stage). Units of equal size are still required at each stage if the overall sample is to be designed for optimum overall costs; however, in case of stratification at the first stage, it is only necessary that the units within a stratum be equal in size.

A little reflection will show that the general model is equivalent to simple cluster work sampling at the final stage with a cluster of size one. When the zero-one variable, $X_{abc\ldots r}$, used in the development of the general model, is replaced by

$$\frac{X_{abc\ldots r}}{S} = \frac{\text{Number of subjects engaged in activity } X}{\text{Number of subjects in the observed cluster}} ,$$

the general estimator becomes

$$p = \frac{\sum_{a=1}^{n_1} \sum_{b=1}^{n_2} \ldots \sum_{r=1}^{n_r} x_{abc\ldots r}}{Sn_1n_2 \ldots n_r} . \qquad (129)$$

In this case the subscript r designates the cluster sampled, and n_r is the number of clusters (of equal size) drawn into the sample. Clusters are completely enumerated as in Chapter V.

The extension to simple cluster sampling at the final stage thus creates no problem in the variance or cost analyses since it involves only a change in the definition of the variable observed and division by a constant. The variance functions will be in terms of $\bar{x}$'s in this case rather than $\bar{p}$'s since the averages are no longer proportions, and the entire analysis is predicated on the basis of a cluster of S subjects being the elementary unit sampled. The cost analysis will follow the notation given in Chapter V, and will be identical to the cost analysis for the general case if simple random sampling of units at all stages is assumed. It should be reemphasized that the clusters are all assumed to be equal in size. Otherwise, some form of ratio estimation will have to be used.

Although it is possible to extend the concept of clustering to the selection of groups of units at stages other than the last, it does not appear that this would be of great practical significance in work sampling. When the units at prior stages are considered as clusters of smaller units (which they are by definition), then the general model may be regarded as consisting of cluster sampling at each stage with clusters being chosen by simple random methods.

It appears more worthwhile to investigate the effects of stratification at one or more stages of sampling. As mentioned earlier, if the basic unit in the population sampled is the activity of a group of subjects rather than the activity of a single subject, stratified random sampling (Chapter IV) at the final stage is the same as stratified cluster sampling (Chapter V) if S is set equal to one. By recognizing this feature of these two methods of sampling at the final stage, we can restrict our analysis to the more general case of cluster sampling in showing the impact of stratification. In this case, an observation made on the activity at a random point in time is a random variable which assumes values between zero and S instead of zero or one, which was the case for the general model.

Because we are still dealing with equal size units at a given level (with the exception that units may vary in size between strata at the first level), the extension of the general model to include stratification is not too difficult in concept. However, the notation becomes rather awkward if stratification is introduced at several stages of the same study. The analysis may be carried out by an extension of the methods shown for stratified sampling at a single stage. The ability to write a model equation for stratification at several stages follows from the fact that variance formulas for the estimator may be built up using the expressions of variance at each level. The construction of such variances, though possible, becomes quite tedious, and are not expected to have wide application in work sampling.

If only the first stage units are stratified and subsequent stages assume simple random sampling, each stratum becomes a separate population to which the methods of the general model can be applied. This approach will yield within strata estimates of the parameters of interest, which, along with their estimated variances, can be combined to give estimates of the population parameters. This concept was introduced in Chapter IV where stratification was discussed in general terms. In the case of stratification at the first stage, an estimate of

$$P_{h'} = \frac{\sum_{a=1}^{N_{1h'}} \sum_{b=1}^{N_{2h'}} \cdots \sum_{r=1}^{N_{rh'}} X_{h'abc\ldots r}}{SN_{1h'}N_{2h'} \cdots N_{rh'}} , \tag{130}$$

the proportion of time spent in state X in the h'th stratum is

$$p_{h'} = \frac{\sum_{a=1}^{n_{1h'}} \sum_{b=1}^{n_{2h'}} \cdots \sum_{r=1}^{n_{rh'}} x_{h'abc\ldots r}}{Sn_{1h'}n_{2h'} \cdots n_{rh'}} ,$$

and an unbiased estimate of P, the proportion of time spent in state X for the entire population, is

$$p = \frac{\sum_{h'=1}^{L_1} N_{1h'} N_{2h'} \cdots N_{rh'} p_{h'}}{\sum_{h'=1}^{L_1} N_{1h'} N_{2h'} \cdots N_{rh'}} . \tag{131}$$

The method used in deriving the variance of the general model can be applied to each stratum to yield the variance, $S^2_{p_{h'}}$, in each stratum. The variance, S^2_p, for the stratified estimator in equation (131) is then the following weighted sum of the strata variances:

$$S_p^2 = \frac{\sum_{h'=1}^{L_1} (N_{1h'} N_{2h'} \cdots N_{rh'})^2 \, S^2_{p_{h'}}}{\left(\sum_{h'=1}^{L_1} N_{1h'} N_{2h'} \cdots N_{rh'}\right)^2} . \tag{132}$$

An estimate of S_p^2 is obtained by using equation (122) to estimate the variance within each stratum and by subsequently applying the weighting factors used in equation (132). The optimum sample sizes for each level of sampling are found by applying the cost analysis of the general model to each stratum separately.

Since it was generally concluded in Chapters IV and V that stratification on the basis of time usually permits a gain in sampling efficiency, it appears that the application of stratified cluster sampling at the final stage of a multi-stage population would be profitable. While the expression for the estimator in this case is straightforward, the analysis of variance approach as previously presented for determining the estimator variance is no longer applicable. Although one can concoct an expression for this variance by introducing the variance of a stratified cluster estimator at the final stage (from Chapter V) and by rewriting the variances at the other stages in terms of this variance, the expressions obtained become quite involved for even a simple case. Rather than deal with such expressions here for the case where units must be equal in size at each stage, we shall turn our attention to the more realistic cases where the units at a given stage(s) are assumed to vary in size.

Variations in Unit Sizes

The variations in sampling schemes indicated in the previous section are important in increasing the sampling efficiency of a study; however, the major restriction placed on the general model is still required in those cases. The restriction was that units at a given stage must be equal in size (at least within strata) for the analysis to be applicable. When units vary in size at any level(s) of multi-stage samples, relatively compact formulations of estimators and their variances similar to the one used in writing the general model are not possible. And since the number of elementary observations which will appear in a sample is not known before the sample is drawn, optimum cost designs in terms of sample size cannot be determined.

Further research in this case will be necessary before straightforward solutions to cost analyses in multi-stage work sampling will be possible if units vary in size. If optimum designs in terms of sample sizes are not required, the multi-stage sampling scheme may still be used to gather and analyze data. This is not as unrealistic as it may seem because estimates of population parameters and their variances are of considerable value in a number of cases when the sample is drawn according to one's judgment as to what would constitute a representative sample. Methods of selecting a sample of units which vary in size may be devised to give unbiased estimates of the parameters of interest as well as estimates of their variances. When historical data are available on costs, variances, etc., within each of the stages, common sense applications of these will lead one to a useful sample design.

Sampling with Probabilities Proportional to Size

Several methods of estimation have been advanced in the survey sampling literature (15, Chapter 11) for treating units of unequal size. As noted in the previous section, if variable size units at a given stage of sampling can be stratified to yield strata within which the units are equal in size, then the methods of that section apply. Otherwise, when the sizes of units are known, as is the case in work sampling, a method of selecting units in which the probability of a unit being included in the sample is proportional to the size of the unit (referred to as pps) has constantly proven to be the most reliable method of sampling (15, p. 206, p. 237). The procedure for this approach in work sampling will be indicated in this section and will be followed by a more simplified procedure for sampling a complex population structure when the components of variance at each stage of sampling are not required.

Consider an example of a three stage population which consists of N departments, N_i work stations in the ith department, and a total of N_{ij} "Subject-instants" in the ijth work station. A sample drawn such that each unit is drawn with pps at each stage will be illustrated using the following definitions:

(1) The elementary unit in the population at the third (last) stage of sampling is a "Subject-instant" denoted by ΔT.

(2) The ijth work station will be composed of a number of elementary units ("instants"), the total of which is

$$N_{ij} = \frac{S_{ij}T}{\Delta T} ,$$

where S_{ij} is the number of subjects in the ijth work station and T is the interval over which the study will be made.

(3) The ith department will be comprised of N_i work stations which make up a total of

$$\sum_{j=1}^{N_i} \frac{S_{ij}T}{\Delta T}$$

elementary units.

There will be no problem in work sampling in determining the exact sizes of the units at each stage of sampling, hence sampling with pps can be achieved using known relative sizes of units. The objective in using pps is to assign an equal probability of selection to each unit in the population. This makes the estimator self weighting and is accomplished by selecting a department (first stage unit) with probability

$$\frac{\sum_{j=1}^{N_i} S_{ij} T/\Delta T}{\sum_{i=1}^{N} \sum_{j=1}^{N_i} S_{ij} T/\Delta T} = \frac{\sum_{j=1}^{N_i} S_{ij}}{\sum_{i=1}^{N} \sum_{j=1}^{N_i} S_{ij}} ,$$

which is the ratio of the total number of subjects in the ith department to the total number of subjects in all departments which make up the population under study.

When department i has been chosen to be in the sample, work station ij is chosen with probability

$$\frac{S_{ij} T/\Delta T}{\sum_{j=1}^{N_i} S_{ij} T/\Delta T} = \frac{S_{ij}}{\sum_{j=1}^{N_i} S_{ij}} ,$$

which is the number of subjects in work station ij divided by the total number of subjects in the ith department. Each subject-instant within the chosen work station has the same probability of selection on each of the n_{ij} observations within that work station. This probability is

$$\frac{1}{S_{ij} T/\Delta T} .$$

Hence, the probability that any given elementary unit, X_{ijk}, will be chosen on a given observation is

P (ith department is chosen) X P (ijth work station is chosen) X

P (ijkth elementary unit is chosen) =

$$\frac{\sum_{j=1}^{N_i} S_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{N_i} S_{ij}} \cdot \frac{S_{ij}}{\sum_{j=1}^{N_i} S_{ij}} \cdot \frac{1}{S_{ij}T/\Delta T} = \frac{1}{\sum_{i=1}^{N}\sum_{j=1}^{N_i} S_{ij}T/\Delta T} ,$$

where the denominator in the last term is the number of elementary units in the entire population.

Since elementary units are chosen with equal probabilities in the case of pps, the sample mean,

$$p = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n_i}\sum_{k=1}^{n_{ij}} x_{ijk}}{m} , \tag{133}$$

is an unbiased estimator for P, the proportion of time in the entire population devoted to element X. In this expression, n is the number of departments in the sample, n_i is the number of work stations from the ith department in the sample, n_{ij} is the number of observations made on the activity at the ijth work station, and

$$m = \sum_{i=1}^{n}\sum_{j=1}^{n_i} n_{ij}$$

is the total number of observations in the study.

To illustrate the foregoing procedure for choosing a multi-stage sample with pps, consider the case where the population of departments (first stage units) and their measures of size are as follows:

CHAPTER VII

DECISION RULES FOR WORK SAMPLE DESIGN

The objectives of this chapter are (1) to state the limitations on the use of work sampling results for predictive purposes, and (2) to develop a procedure whereby one may make a rational decision as to which of the foregoing sampling methods should be employed in a given work sampling situation. The limitations are dependent on the amount of variation in the process being studied, whereas the optimum method of sampling is a function of the population structure.

Process Variability and a Definition of the Population

Fundamental to the design of a work sampling study is a definition of the population to be sampled. A statement of the activity of interest and the determination of the exact period of time to be sampled constitute definition of the population and result from a desire either to measure the characteristics of the activity over the chosen period or to predict values of population characteristics for future periods. In the first case, there is no problem; e.g., if the objective is to determine the fraction of time spent in the various categories of an activity for a given month, and if the month in question is designated as the population, then the estimators defined in the previous chapters of this study are applicable. In the second case, the problem is more difficult, and the results of a study must be regarded only as approximations for reasons which we shall discuss.

Departments	Subjects	Cumulative Sum
1	33	33
2	90	123
3	71	194
4	27	221
5	60	281
∘	∘	∘
∘	∘	∘
∘	∘	∘
N	S_N	$375 = S^*$

Each of these departments is assumed to be made up of a number of work stations, the detailed composition of which is not needed at this stage.

In order to determine the sampling procedure at each stage, it will be necessary to decide beforehand the intensity with which one wishes to sample, e.g., one observation for each ten man-minutes of activity in the population, as well as the average number of observations desired in an "ultimate cluster." An ultimate cluster is a group of N_{ij} observations from the jth work station within the ith department. These decisions require a general knowledge of the population and will reflect the decision maker's personal judgment as to what will be a "representative" sample. The resources allocated to the conduct of the study will very likely fix the overall sampling intensity. For example, the number of observers available may result in the limitation that, on the average, one observation per ten minutes of activity is the maximum sampling intensity that can be achieved without additional personnel. We shall assume that such an analysis has been made and that an overall sampling intensity of one observation for each twenty man-minutes of action is indicated.

Since the observations within an ultimate cluster will be a sample from an infinite population of instants, we shall assume that the desired sample per ultimate cluster, $\bar{n}_{ij}$ is fixed before the study begins at a selected level of 200 observations, say. In addition, we shall assume that the study will cover ten working days or 4800 minutes. With this information, we may proceed with the determination of which units will be drawn into the sample at each stage of the population. It should be noted that the data required for this procedure do not need to be refined in order for pps to work. The choices of $\bar{n}_{ij}$ and the sampling intensity are needed in order to approximate the desired coverage of the sample, and whatever arbitrariness exists in their selection will in no way invalidate the statistical evaluation of the data. We choose the ultimate clusters within departments by the following procedure:

(1) Determine the number of ultimate clusters to be included in the sample as five per cent of the total man-minutes in the population divided by 200. For the illustration this will be $(0.05)(4800)(375)/(200) = 450$.

(2) Choose 450 random numbers between zero and S^*, the total number of subjects in the population, and designate the departments chosen by checking these numbers in the cumulative sum column of the foregoing tabulation. These numbers are drawn with replacement since more than one ultimate cluster is allowed per department.

(3) The procedure in (2) will designate the departments to be included in the sample and the number of ultimate clusters from each. These departments (and only these) will now be enumerated and listed in terms of work stations and their sizes. For example, assume that depart-

ment three was the first on the above list to be included in the sample. We would list it for further sampling as follows:

Department 3:

Work Station	Number of Subjects	Cumulative Sum
1	3	3
2	5	8
3	1	9
4	2	11
5	6	17
6	4	21
.	.	.
.	.	.
.	.	.
N_3	8	71

If the above department was designated for eight ultimate clusters, we would choose eight random numbers between zero and 72 and designate which of the above work stations would be included in the sample. Again, in order to keep the sampling pps, we allow a work station to be chosen more than once. Repeat this procedure for each department designated to be in the sample. The end result will be the complete designation of all ultimate clusters to be drawn into the sample.

(4) With the work stations designated for sampling in each department, we proceed to make the actual observations on the activity of interest. If a work station is chosen n^* times in (3) above, then the sample from this work station will be a total of $n^*\bar{n}_{ij} = 200n^*$ observations. Each ultimate cluster will be drawn using simple random sampling within each work station by using the procedures set forth in Chapter III of this study. Stratification may also be introduced into the scheme by stratifying on time over the period of the study, if the

same stratification is used in the collection of data in each ultimate cluster. In this case, the procedures of Chapter IV are followed within each ultimate cluster. It is also noted that the entire population of first stage units may be stratified at the outset and the above procedure applied to each stratum. The resulting stratum estimates and variances may then be combined for an overall estimate.

Using the foregoing procedure, it is easy to see that the sample is self weighting and that the sample mean is an unbiased estimate of the population mean. The self weighting feature follows from the fact that each unit in the total population has the same probability of being in the sample for each observation that is made. It is convenient for the purposes of calculating the variance of the overall estimate, p, to record the data in terms of ultimate clusters, i.e., when a work station is chosen n^* times, to keep the data separated into n^* independent sets of $\bar{n}_{ij}$ observations each.

The estimator in equation (133) was

$$p = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{n_{ij}} x_{ijk}}{m}$$

and is simply the average of the sample, as noted previously. In order to facilitate the variance calculation of p, we shall extend this expression to indicate ultimate clusters as follows:

$$p = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{\ell=1}^{n_{ij}^*} \sum_{k=1}^{\bar{n}_{ij}} x_{ij\ell k}}{200 \sum_{i=1}^{n} \sum_{j=1}^{n_i} n_{ij}^*} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{\ell=1}^{n_{ij}^*} p_{ij\ell}}{\sum_{i=1}^{n} \sum_{j=1}^{n_i} n_{ij}^*}, \tag{134}$$

where k denotes the kth observation in the ℓth ultimate cluster in the ijth work station, and $p_{ij\ell}$ is the average of the 200 observations in the $ij\ell$th ultimate cluster. There are n_i work stations in the sample from the ith department, and a total of n departments are in the sample.

The variance of p may now be written in a simple fashion as the variance between the $p_{ij\ell}$ within ultimate clusters. The expression for the variance estimate is

$$s_p^2 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{\ell=1}^{n_{ij}^*} (p_{ij\ell} - p)^2}{\left\{\left(\sum_{i=1}^{n} \sum_{j=1}^{n_i} n_{ij}^*\right) - 1\right\}\left\{\sum_{i=1}^{n} \sum_{j=1}^{n_i} n_{ij}^*\right\}}. \tag{135}$$

Other variance estimates may be calculated from the data, e.g., the variance between departments, the variance between work stations within departments, etc., if the data are properly identified during collection. Although these may be of occasional interest, the main estimators of concern are those of the overall population parameters, which are easiest to obtain by the procedure. The estimator p of the proportion of time spent in state X in the entire population, and its estimated variance, may be used for setting confidence interval estimates on P by assuming normality as in the previous models.

A Method for Eliminating the Problem of Unit Size

Sampling with pps allows a sample design which leaves the population structure intact by letting the unit sizes affect the selection procedure. Deming (21) and Tukey (20, p. 93) have devised ingenious schemes for circumventing the problem of calculating estimates and their variances in similar multi-stage populations by restructuring the population such that equal probabilities of selection may be used with replicated samples. The restructuring of the population into "paper zones" for the purposes of sampling is due to Deming and was motivated by Tukey's use of replicated samples in surveys. These two concepts may be adapted to work sampling problems for the purpose of yielding an estimate of population proportions and their variances. The resulting estimators and their variances are expressed in the more simplified notation of single stage sampling on units of equal size, properties which were noted at the outset of this chapter to be very desirable.

To illustrate the method, consider the population structure presented previously in connection with the pps concept. In order to eliminate the problem of variable size units, the population is listed such that it can be divided into zones of equal size as follows.

(1) List the entire population by work station and work station size, being careful to list work stations of like activity together. For the illustrated case, this would appear as follows:

Factory	Work Station	Number of Subjects	Cumulative Sum	
1	1	3	3	
	2	5	8	Zone 1
	3	1	9	Z Subjects
	4	6	15	
	5	3	18	
	.	.	.	Zone 2
	.	.	.	Z Subjects
	.	.	.	
	N_1	4	33	
2	1	4	37	
	2	5	42	.
	3	3	45	
	4	2	47	
	5	7	54	
	.	.	.	.
	.	.	.	.
	.	.	.	
	N_2	6	123	
3	1	4	127	.
	2	3	130	
	.	.	.	
	.	.	.	Zone m
	.	.	.	Z Subjects
	.	.	.	
	.	.	.	
N	N_N	5	S^* = Total Employees	

The practice of listing like work stations together serves to create homogeneous strata when the zones are formed and may be extended to listing work stations together from throughout the population when the variation between such work stations is significant.

(2) Using the above table and knowledge about the groups of like work stations listed consecutively, one chooses a zoning interval, Z, which will divide the entire population into groups of Z subjects. The width of this zoning interval is not critical, although the better it separates the subjects such that the activity within zones is more homogeneous, the greater will be the gain from stratification. Since replications (multiple samples) will be drawn from within

zones and used to estimate the population parameters of interest, the greater the number of zones, the more degrees of freedom the variance estimates will have. However, too many zones would create administrative problems as well as increase the amount of sampling unnecessarily. The formation of zones must be a judgment decision, but it can be handled in a fairly efficient fashion if the practitioner familiarizes himself with the population.

(3) The creation of zones is carried out as indicated in the right hand column of the above table by marking off groups of Z successive subjects. These zones are equal in size and contribute equally to the estimates of population parameters and their variances.

(4) To capture the gain from stratification (differences between zones) and to allow an estimate of variance, a random sample of k subjects is chosen from within each zone and n observations are made on each subject over the period of the study. The sample may be drawn using either the simple random sampling methods of Chapter III or stratified sampling on time as in Chapter IV. If stratification on time is used, the same time strata should be used for each subject chosen in the sample in order to keep the estimators the same in all zones.

The k independent estimates of P from each zone afford a means of calculating the variance within zones. If p_{ij} is the estimate of P_{ij} from the sample on the jth subject in zone i, an estimate of the variance of p_{ij} is as follows:

$$s^2_{p_{ij}} = \left(\frac{Z-k}{Z}\right) \frac{\sum_{j=1}^{k} (p_{ij} - p_i)^2}{k-1}, \tag{136}$$

in which p_i is the overall estimate of P_i from the k samples in the ith zone.

Due to the fact that zones are equal in size, simple expressions for estimating P, the proportion of total time that the entire population spends in state X, and their variances, may be written as follows:

$$p = \frac{\sum_{i=1}^{m} p_i}{m} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{k} p_{ij}}{mk} , \tag{137}$$

$$s_p^2 = \left(\frac{Z-k}{Z}\right) \frac{\sum_{i=1}^{m} \sum_{j=1}^{k} (p_{ij} - p_i)^2}{km^2(k-1)} .$$

These simple formulas replace the complex formulas associated with multi-stage sampling in general and allow a relatively easy procedure for controlling variable unit sizes. Although the variance formula for sampling by zones is similar to that for sampling with pps, the sample design is much simpler in the former case since all sampling is with equal probabilities. The sample is actually distributed over departments, factories, etc., in the same fashion that pps distributes it since a given factory will contribute a number of zones proportional to its size.

The freedom which one has in stratifying the population in the basic design, both in terms of zones as well as with respect to time, adds to the significance of the zoning scheme. The estimator precision may be adjusted by varying the number of replications chosen from each zone as well as by varying the number of observations per replication. This feature of the estimate of variance may be widely exploited by the practitioner, depending on its sensitivity in each particular case. It is such features as these that a more precise work sampling methodology must have if it is to be of much practical value to the general practitioner.

Limitations of Multi-Stage Work Sampling

As pointed out in the introductory remarks of this chapter, multi-stage work sampling is a method for collecting data from very broad systems of activity. Because of this, its primary use will be at corporate level studies. In these large scale populations which may be spread over several geographic locations, the methodologies presented in this chapter afford a sampling technique which is far superior to the simpler methods presented in the earlier chapters. In those cases, the population was a well defined period of activity at one location, any part of which could be observed without extensive travel. The application of those methods to a population which is large in size and/or not located at the same place requires excessive administrative detail and cost and should be avoided for this reason.

As already noted elsewhere in this chapter, optimization schemes for controlling estimator variances and/or overall study costs are not available unless population units are all equal in size. This is a severe limitation of the method and will preclude, in general, the design of a study on the basis of rigid cost considerations. It is noted, however, that the application of the methods of zoning and sampling with pps, as herein set forth, may be used efficiently in sampling large populations if the practitioner makes full use of historical data on costs, variances, and other characteristics of the population which may be available to him.

The estimators from any sampling scheme pertain only to the population from which the samples are drawn. Hence, they are no better in providing the desired results than the sufficiency of the designated population in encompassing the population about which information is desired. Because of this fact, it is not possible to make absolutely correct predictions about what will take place during some future period of an activity by studying some previous period of that activity. It is in this respect that work sampling differs rather markedly from survey sampling. In the latter case, the population sampled is a physically tangible set of units which may vary slightly with time, but which, for the purposes of sampling, may generally be considered as a stable population. In the former case, the sampling units are segments of time and, as such, are not capable of being resampled, or restudied, as is the case in survey sampling. With this "fleeting" aspect of the units in work sampling, one can only sample a population as it comes into existence. The characteristics of an activity over a given period of time (a work sampling population) may be estimated, but unless the subsequently defined populations are similar to that sampled, the estimators obtained previously are not applicable.

There are numerous instances in which interest is concentrated on a defined period of the activity. In these cases, the estimators as developed in earlier chapters and the precision placed on the results in each case are completely valid for estimating the characteristics of such a period. On the other hand, many work sampling studies are conducted for the purpose of determining characteristics of a continuous activity, such as, for example, the amount of idle time in a machining department, or the fraction of time executives spend on the telephone. In estimating these characteristics, one must be careful in making inferences about a segment of the activity which is broader than that sampled. This is true because the sample is from a different population than the one about which information is sought. When there is known variability from period to period, a continuous program of study involving work samples over consecutive periods of time provides a means for estimating the period-to-period variability.

The foregoing paradox is only partially resolved in actual work sampling applications. If the continuous activity is cyclic, then inferences may be made about the total overall span of the activity if the work sampling population is defined as some multiple of the cycle. The assumption of cyclic behavior is seldom, if ever, absolutely valid. The many independent sources of variation in the performance of most activities account for this dynamic aspect of populations defined in time. However, it has been repeatedly observed and also appears logical, that the major sources of variation in many activity populations tend to be cyclic in nature. This accounts for the successful assumption in the past that work sampling results may be used to predict future performance.

It is conceivable that the dynamics of some activities may be defined as stochastic processes and accordingly may be analyzed in terms of probability measures. While eventually this may be accomplished, the complexity of the variations in the types of activity typically studied in work sampling prohibits an accurate description of them by the general theory of stochastic processes thus far developed. Therefore, it is assumed that the application of any of the previously defined models to predict the characteristics of some future population implies the assumption of cyclic or stable population behavior. Otherwise it will be assumed that a continuing program of work study is carried out and that the dynamics of the population will be estimated by the variability in estimates of the characteristics from consecutive studies. In any case, from a technical viewpoint, the estimates pertain only to the population that was sampled.

The Choice of a Work Sampling Plan

As the various work sampling models were developed in the earlier chapters of the present study, discussions of each of them included the unique characteristics peculiar to that specific design. The two major concepts presented beyond that of simple random sampling were those of stratification and clustering. The unique characteristic of multi-stage sampling was the idea of sampling smaller and smaller units, with each successively defined unit being a subdivision of the previous one, and each division of the unit representing a separate stage of sampling. The theory in the multi-stage case is a combination of the three previous models -- random, stratified, and cluster -- in which one or more of these concepts are employed in designing sampling schemes for the efficient study of total systems of activity on a broad scale.

It becomes readily apparent that one can choose a sampling scheme for a given situation from among those available by answering, either in the affirmative or the negative, a few basic questions. These questions and the order in which they should be answered are:

1. Does the total scope of the activity cover a large geographic area, and/or is the "natural structure" of the activity such that it is

made up of large divisions of activity within which there are smaller divisions? There may be only one such division or there may be several.

If the answer to (1) is yes, then multi-stage sampling is appropriate and (2) and (3) below are asked with respect to each level of sampling when the population consists of equal size units. If the population cannot be defined in terms of units of equal size, the methods of pps or zoning are appropriate. If the answer to (1) is no, then (2) and (3) are asked with respect to the total population. In this case, the answers to the last two questions will result in the specification of the proper sampling scheme.

2. Is the activity of such a nature that the subjects in question can be sampled more advantageously in groups than singly?

If the answer to (2) is yes then a cluster model will be in order, otherwise a model which requires observations on a single subject will be used. In either case, the answer to (3) completely specifies the model to be used.

3. Is it possible to subdivide the total population to be sampled into any type of segments such that the proportions of interest will be significantly different between segments?

If the answer to (3) is yes, then the model to be employed will be a stratified model. Otherwise, the model will employ simple random sampling. These decisions are better illustrated by the decision-diagram in Figure 9.

If the design is multi-stage, then one of the procedures set forth in Chapter VI is followed for establishing the estimator, its variance, and, in the case of equal size units, the optimum amounts of

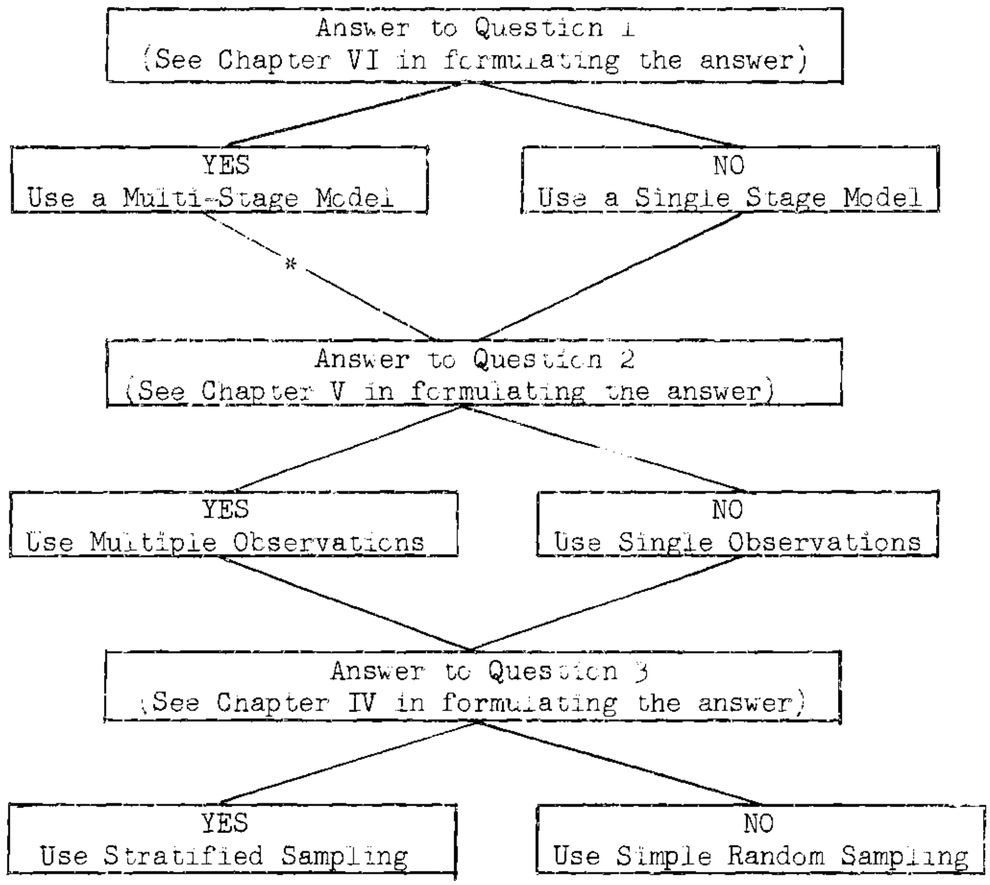


Figure 9. Decision-Diagram for Obtaining the Proper Work Sample Model in a Given Application.

sampling at each stage. The latter case will involve evaluating the components of variance at each level of sampling using the general multi-stage model. However, it is expected that units will vary in size in most large studies and hence render this approach inappropriate. The alternatives of pps and zoning must then be considered in arriving at the final sampling scheme. The choice between these two methods cannot be stated categorically since they are not in direct competition in

*The remaining questions are repeated for each stage of sampling if units of equal size are to be sampled. In the event of units of unequal size, the methods of pps and zoning are applicable.

terms of efficiency. The relative merits of the two procedures are set forth in Chapter VI where it is concluded that the technique of zoning will generally be favored. In the cases where multi-stage sampling does not apply, the single estimators, their variances, etc., as presented in Chapters III through V yield the correct model immediately.

It should be noted again that in every case where a cluster estimator is used, if S, the number of subjects, is set equal to one, the resulting model is either simple random or stratified random sampling. The decision to present these two approaches as separate models, when in fact one is a special case of the other, results from the ability to write the estimator and its variance in much simpler form when $S = 1$.

In view of the simplicity of the foregoing rules and the limited amount of information which they require, it appears that the work sampling practitioner should have little difficulty in resolving the question of which sampling model will be the most efficient in any given situation.

CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

The major objective of this study was to improve the sampling procedure and subsequent data analysis for work sampling studies in an effort to yield more reliable estimates of time spent in various categories of the activity being analyzed. The procedure for accomplishing this objective was to develop a new theoretical basis for work sampling in the form of categorized sampling models, incorporating provisions for assessing the nature of the population being sampled and for specifying the theoretical model which would best accommodate cost and reliability constraints.

The conventional method of work sampling using simple random sampling and binomial probabilities has been analyzed in detail and presented as the first of four proposed models. An analysis of the error introduced when the sample variance is used to approximate the true variance in work sampling has resulted in a graphical means of determining the sufficiency of the variance estimate. An explicit justification for using an "instant" of time as the sampling unit has been made and detailed graphs useful in designing a simple random sample in terms of both relative and average error have been devised.

Simulation has provided data to support the assumption of normality for the distribution of the estimator, and for the verification of the simple random work sampling model in general.

An analysis of the nature of stratification in work sampling populations has resulted in the development of a stratified work sampling model. Comparisons of the variance of stratified estimators with those of simple random estimators have shown that the sample sizes necessary for a stated level of precision may be considerably smaller in the former case. Investigation of the problem of allocating a total sample to a number of strata has shown that the differences between sampling costs from stratum to stratum and the differences of within strata variances have little influence on the problem of optimum work samples. As a result of this finding, and because of the administrative convenience of taking a sample from each stratum in proportion to the size of the stratum, the stratified model presented herein assumes proportional allocation.

The gain in reduced sample size by using a stratified sample rather than a simple random sample is shown for a hypothetical case by simulated sampling of the same population used to support the simple random model. The results of the simulation show that the stratified estimator may also be assumed to follow a normal distribution. The conflict between achieving minimum costs and stated levels of precision in sample design is illustrated for stratified work sampling as well as for the other models.

The practice of making group observations in work sampling is analyzed in terms of cluster sampling theories and practices from the field of survey sampling. The nature of the error introduced by making the assumption of independence among subjects in groups is shown, and a method of analysis which is free of this assumption is advanced. The validity of cluster work sampling is also illustrated through simulated sampling of the same hypothetical population used for illustrating the other models.

It was shown through the development of a cost model for cluster work sampling that, although more observations are required for a stated precision than in simple random sampling, the overall costs of cluster sampling are generally less. This realization results from the fact that observations made in clusters are less costly on a per observation basis.

The concept of stratification was coupled with that of clustering to provide a model which is more efficient than either of the models employing these concepts singly. It is concluded that the model for stratified cluster work sampling is the most efficient of all the models which are applicable to the study of activities confined in such a way that travel costs between observations are insignificant.

A work sampling model for sampling activities which are not so confined, e.g., they may encompass several geographic locations, is developed by employing the concept of sampling in stages. The multi-stage model which is presented requires that all units at a given level of sampling be equal in size and that simple random sampling be used at each stage. This restriction makes the model useful for little more than a point of departure in discussing more realistic situations. The deviations from the general model which are analyzed fall into the categories of variations in sampling scheme and variations in unit sizes. The latter are of most importance and two sampling schemes are presented for use in this case. The first of these uses the concept of an ultimate stratum and allows a unit to be drawn into the sample with a probability which is proportional to its size. The second method offers a desirable scheme which permits stratification as well as sampling with equal probabilities. The key to this latter procedure lies in the restructuring of the population into "paper zones" for the purpose of choosing a sample.

A general discussion of process variability and the use of work sampling data for predictive purposes point out the shortcomings as well as the usefulness of work sampling data. The study is concluded with a presentation of a set of decision rules for choosing the proper sampling scheme in a given situation.

In general, this study has shown that most work sampling populations may be sampled more efficiently than by simple random methods by exploiting certain easily observed characteristics of the population structure. The magnitude of the gains in efficiency has been illustrated in terms of both precision and overall study costs. Pertinent costs and their influence on the problem of optimization have been enumerated and analyzed through the formulation of cost models for the sampling schemes which have been proposed. The application of these methods by work sampling practitioners will result in more useful, economical, and reliable data for subsequent managerial decisions.

In the course of this study, a number of problems arose which were beyond the scope of the study; however, some of these do have an influence on the problems treated herein. Those which are significant enough to warrant further study follow.

This study was concerned with the extension of work sampling practices by the use of methods of survey sampling. These extensions necessitated the consideration of costs in work sampling in a general way, which were treated in terms of cost models for each sampling scheme. Based on the findings in these cases, it is recommended that the components of costs as herein treated, especially the component of "losses due to error," be investigated to determine their nature and magnitude. In the latter case, it is anticipated that sets of loss functions could be determined such that the work sampling practitioner could choose a function by asking the decision-maker a few easily answered questions and translating the answers in terms of cost curves. Since this loss is a function of the error in the estimate, it is suspected that most practical cases could be represented by a set of curves which would not need to be too extensive in scope. This research would deal mainly with the determination of the parameters of such curves by using data which could be obtained from firms which use work sampling.

Additional research should be directed to the solution of cost optimizations in the cases of varying cluster sizes and multi-stage sampling involving units of unequal sizes. While these have been treated superficially in the literature, no explicit solutions appear to be forthcoming due either to the uncertainty of the size of sample which will be drawn or to the inability to determine certain costs and variances.

Due to the varying nature of many activities to which work sampling is applied, and to the subsequent inability to define a population which will be representative of future periods of time, the concept of sequential sampling should be investigated to determine whether or not an analogous scheme can be devised for determining when to stop sampling such an activity.

Finally, studies which would deal with the problem of implementing these new sampling schemes into current practice are certain to be fruitful. Methods of presentation, including films, slides, or other visual aids for making the concepts easily understood by practitioners might be the aim of such a study.

APPENDIX A

HYPOTHETICAL WORK SAMPLING POPULATION

The hypothetical population depicted in this appendix consists of four subjects engaged in an activity comprised of three elements (or states). These elements are identified as follows:

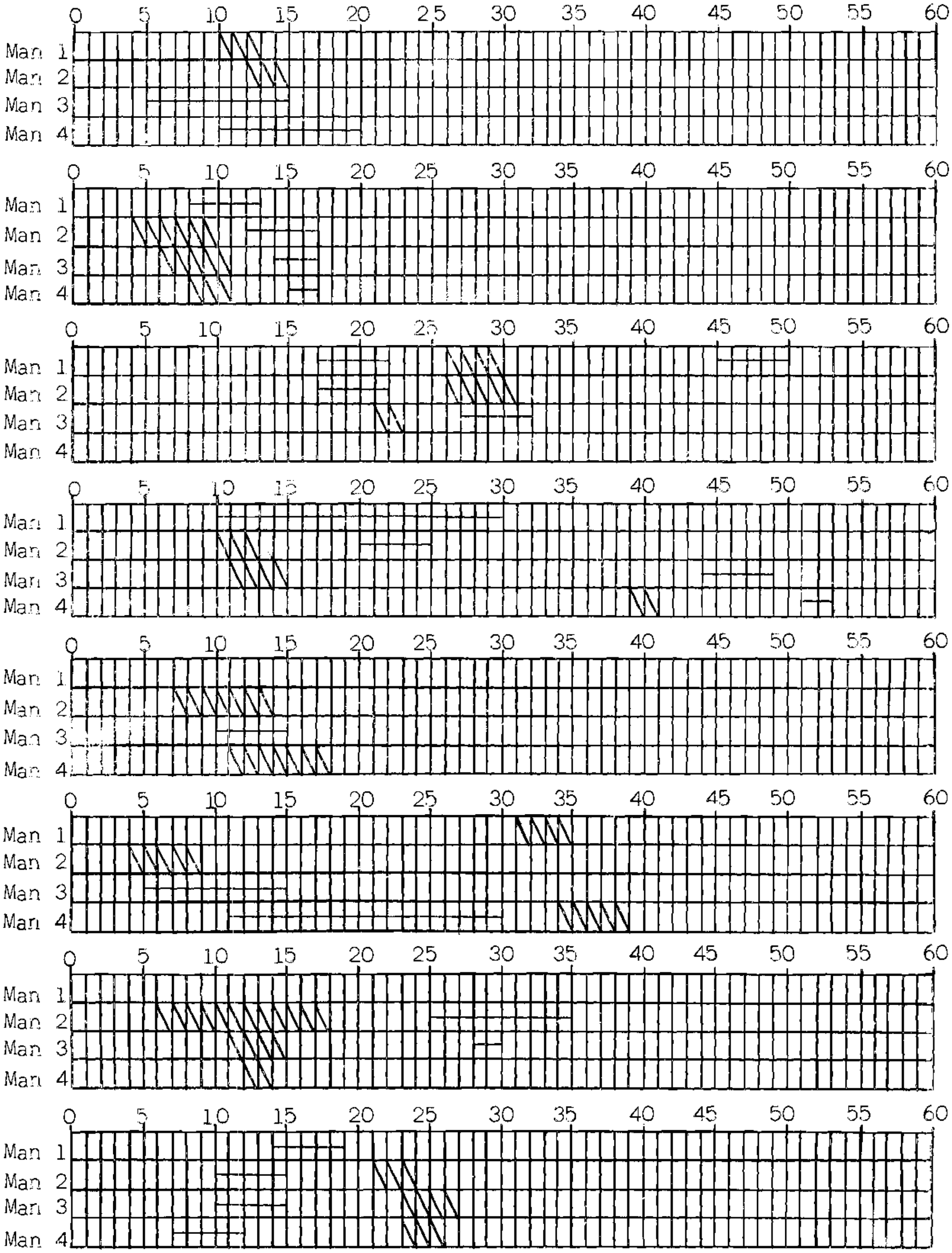
- Element 1, $P_1 = 0.1650$

- Element 2, $P_2 = 0.2570$

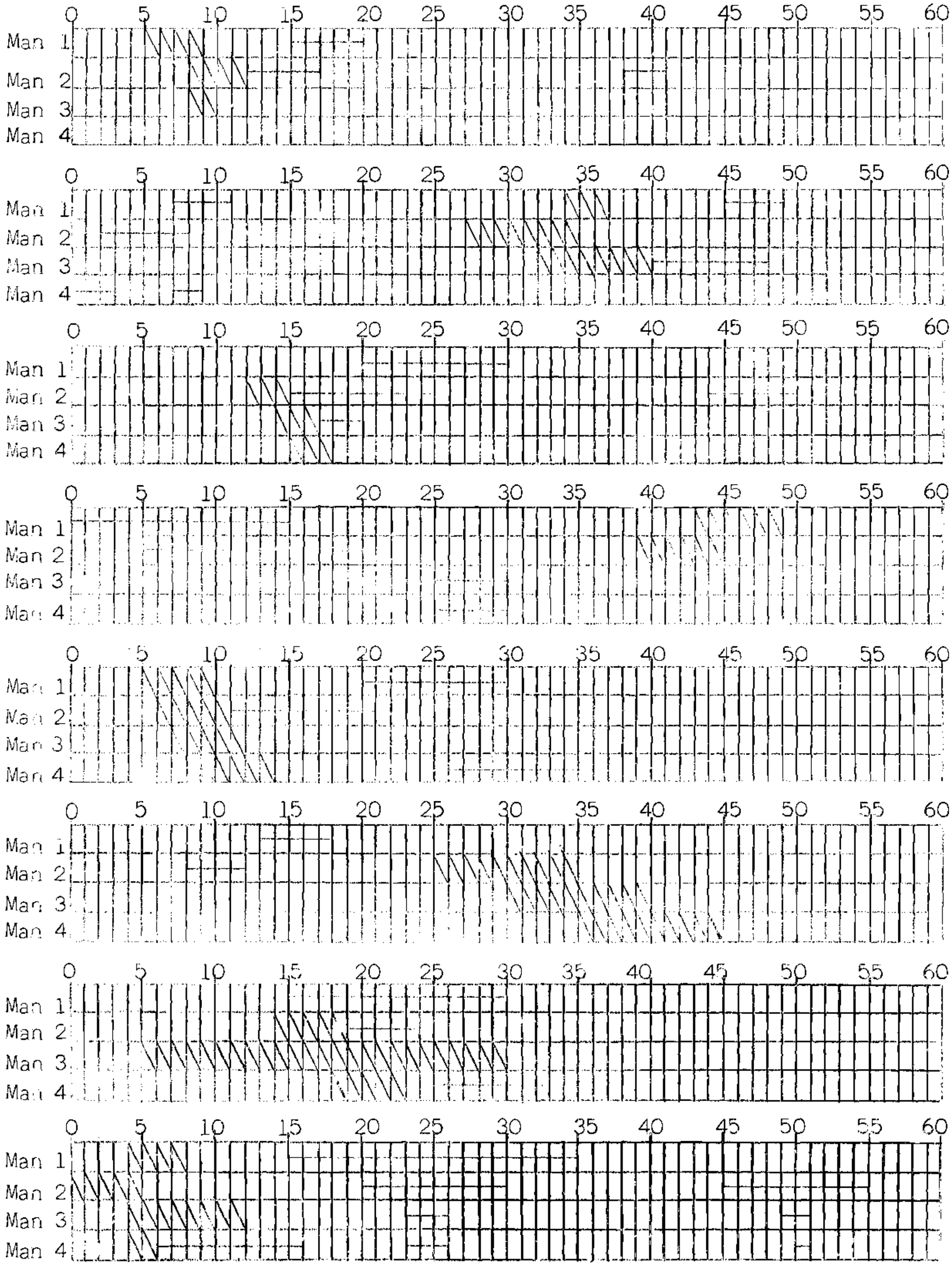
- Element 3, $P_3 = 0.5780$.

The overall length of the activity is 4800 consecutive clock minutes, the equivalent of ten regular work days. The assignment of an element to each of the 4800 x 4 = 19,200 man-minutes has been done arbitrarily, which in no way affects the use of the appendix in the study. The distribution of the man-minutes occupied by each of the elements over days, hours, and subjects was carried out in a manner which illustrates the concepts of stratification and clustering discussed in Chapters IV and V.

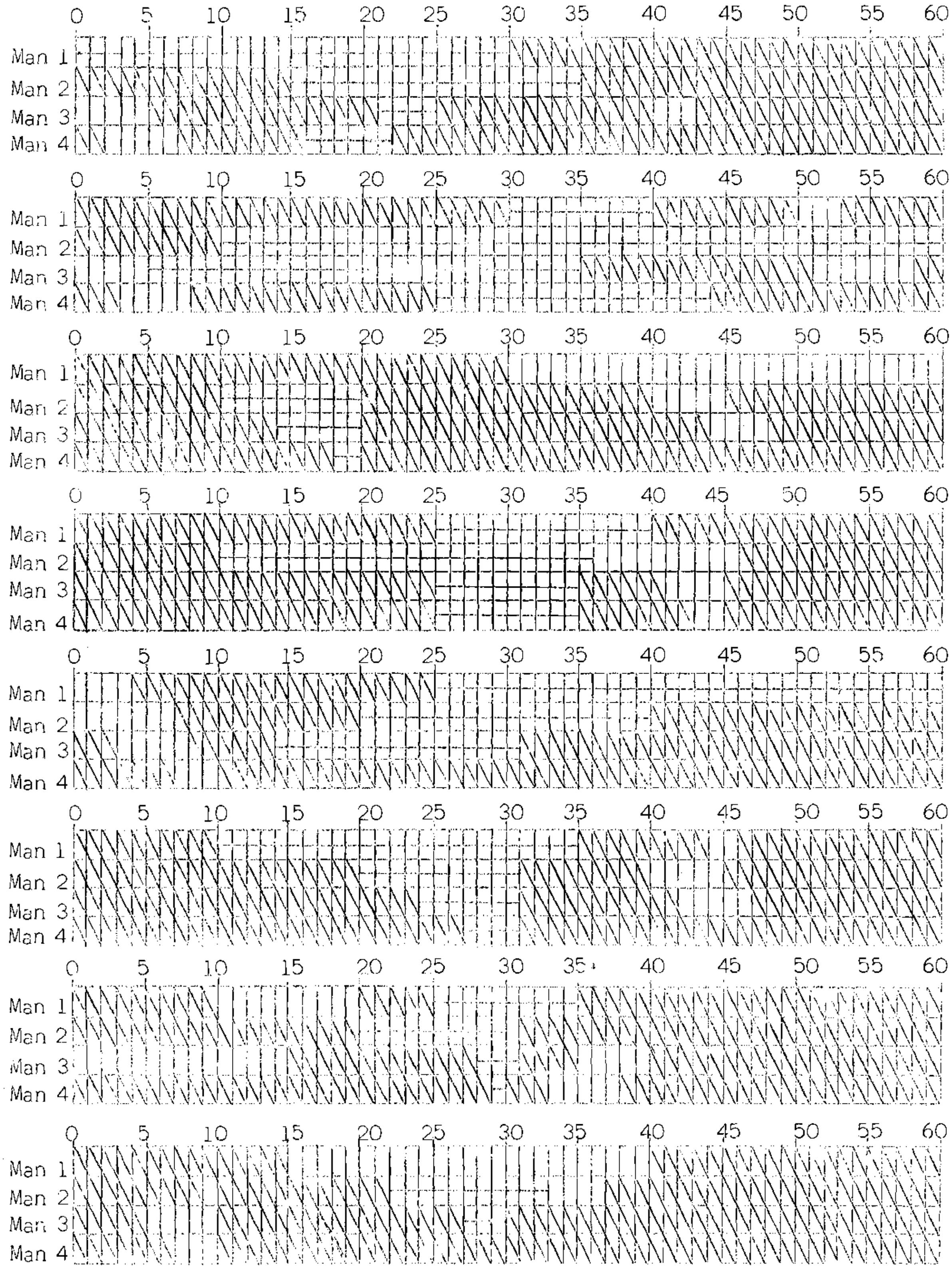
ACTIVITY ARRAY IN MINUTES FOR WEEK 1, DAY 1, HOURS 1 - 8



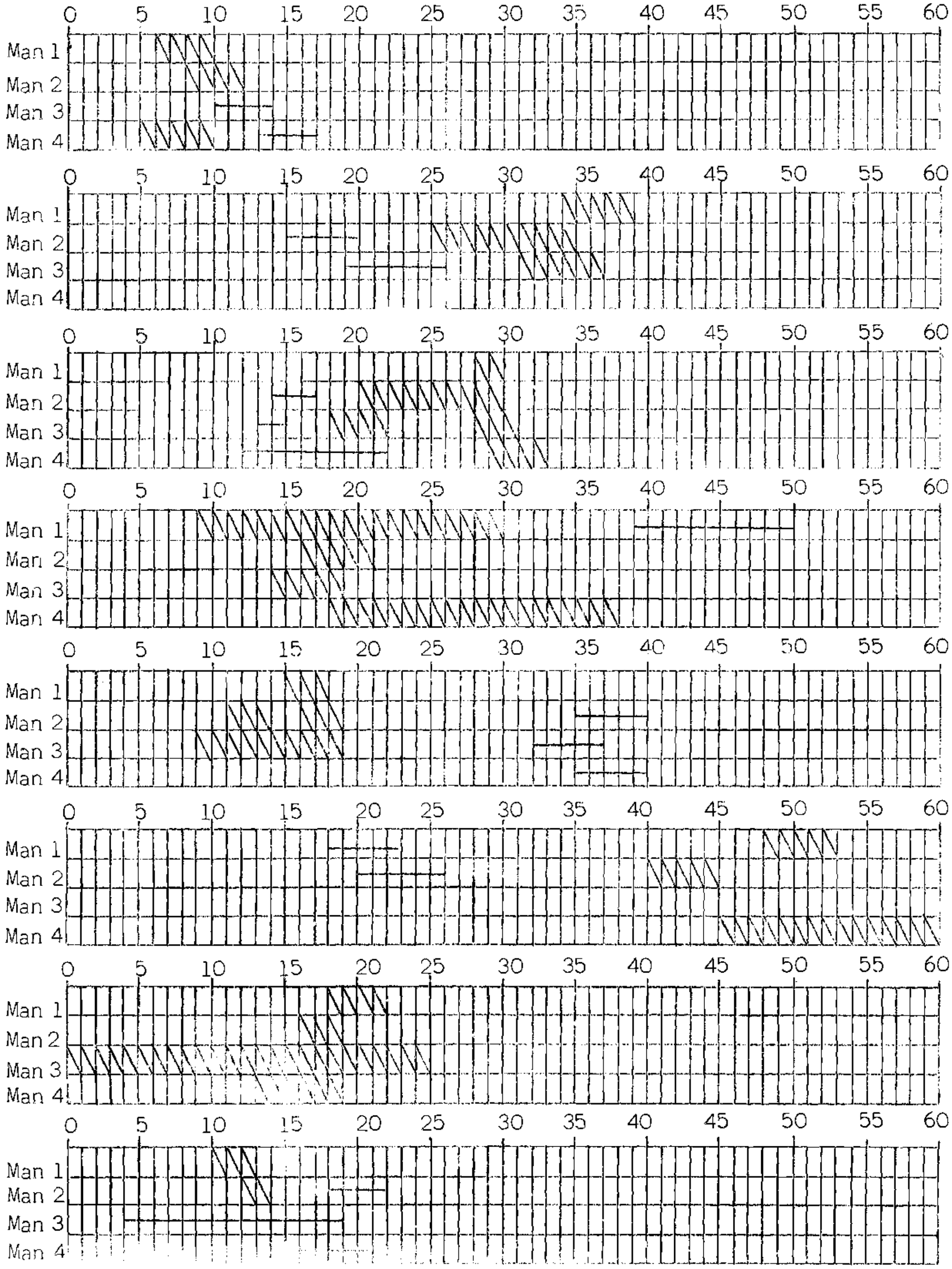
ACTIVITY ARRAY IN MINUTES FOR WEEK 1, DAY 2, HOURS 1 - 8



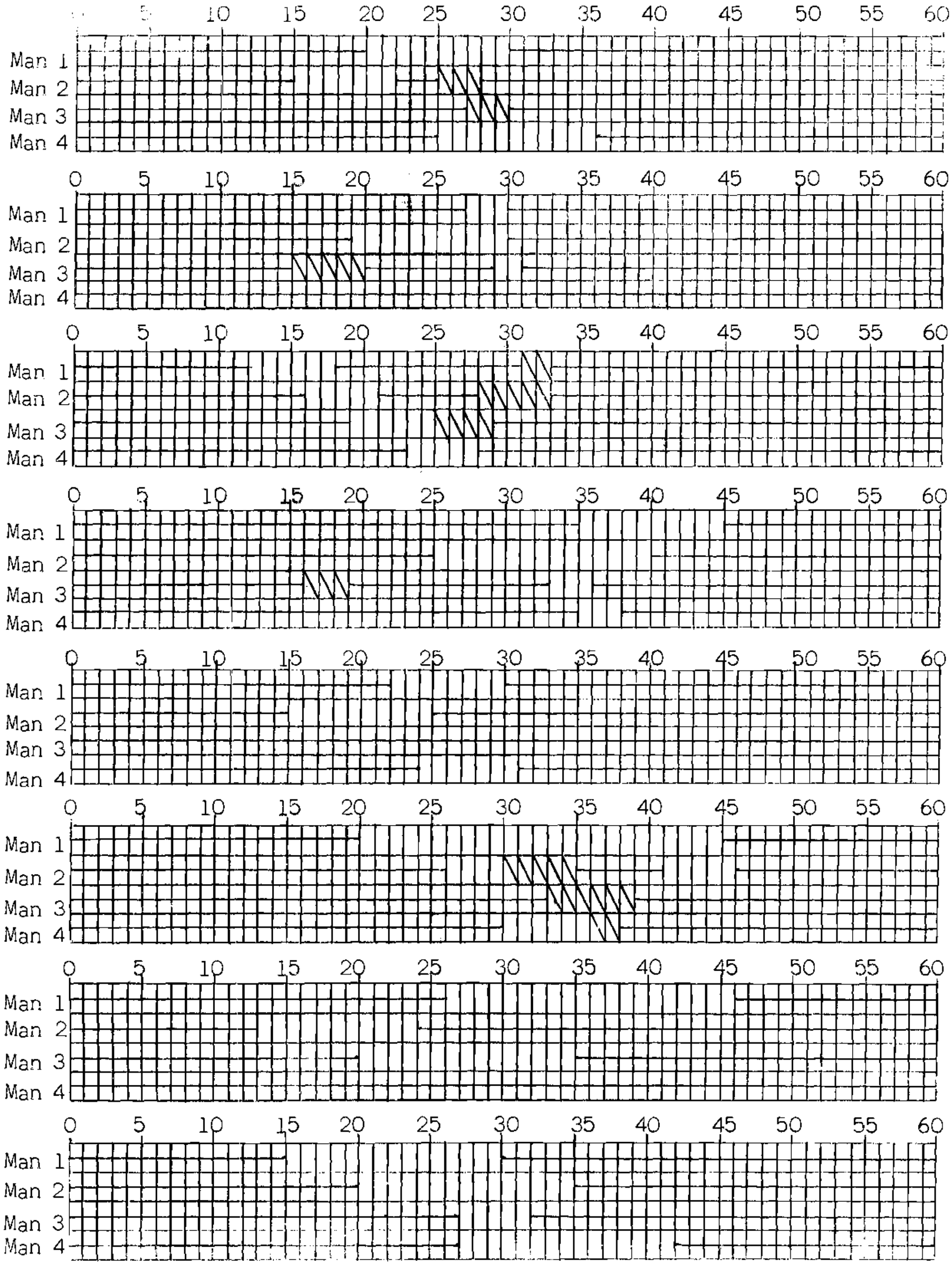
ACTIVITY ARRAY IN MINUTES FOR WEEK 1, DAY 3, HOURS 1 - 8



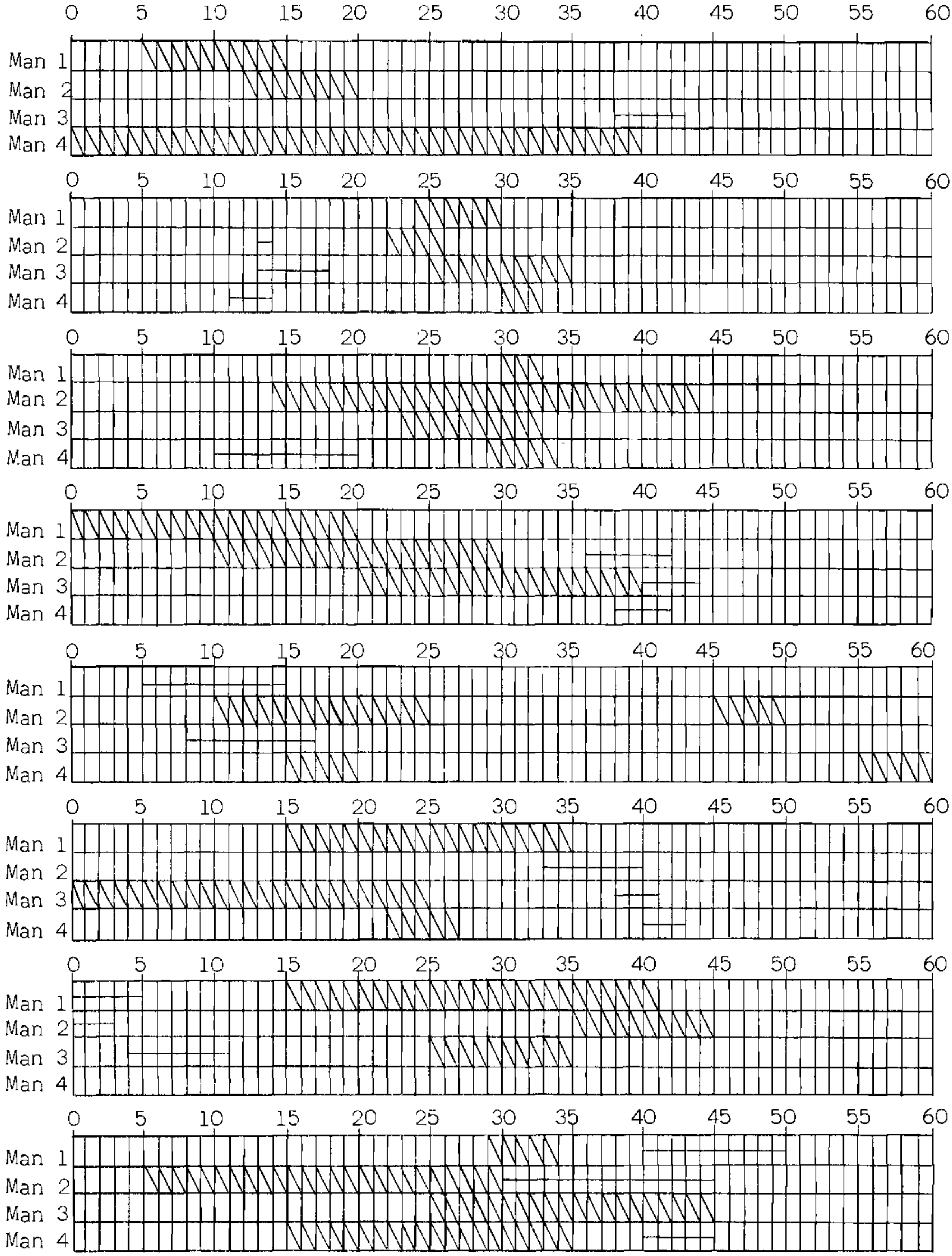
ACTIVITY ARRAY IN MINUTES FOR WEEK 1, DAY 4, HOURS 1 - 8



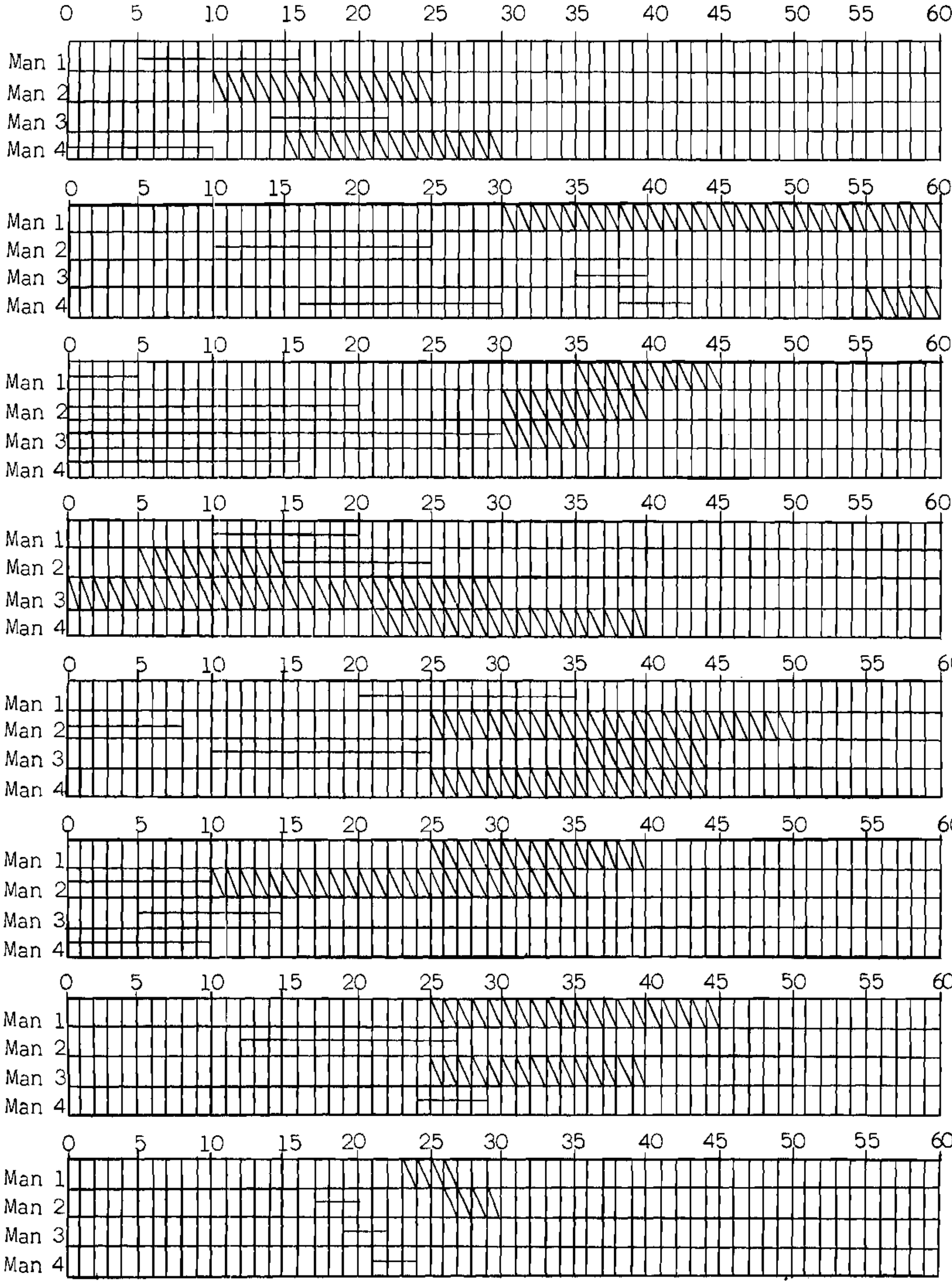
ACTIVITY ARRAY IN MINUTES FOR WEEK 1, DAY 5, HOURS 1 - 8



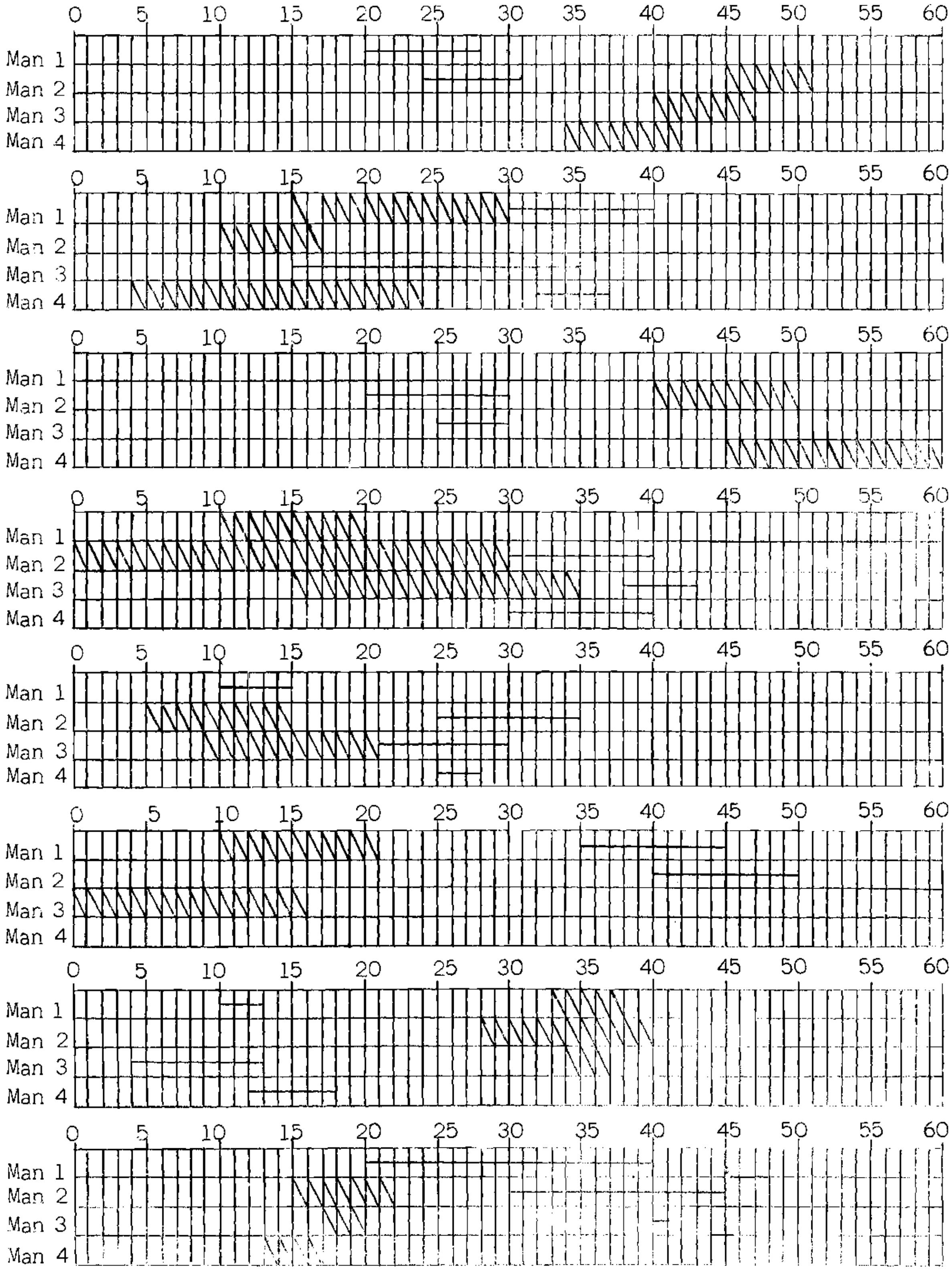
ACTIVITY ARRAY IN MINUTES FOR WEEK 2, DAY 1, HOURS 1 - 8



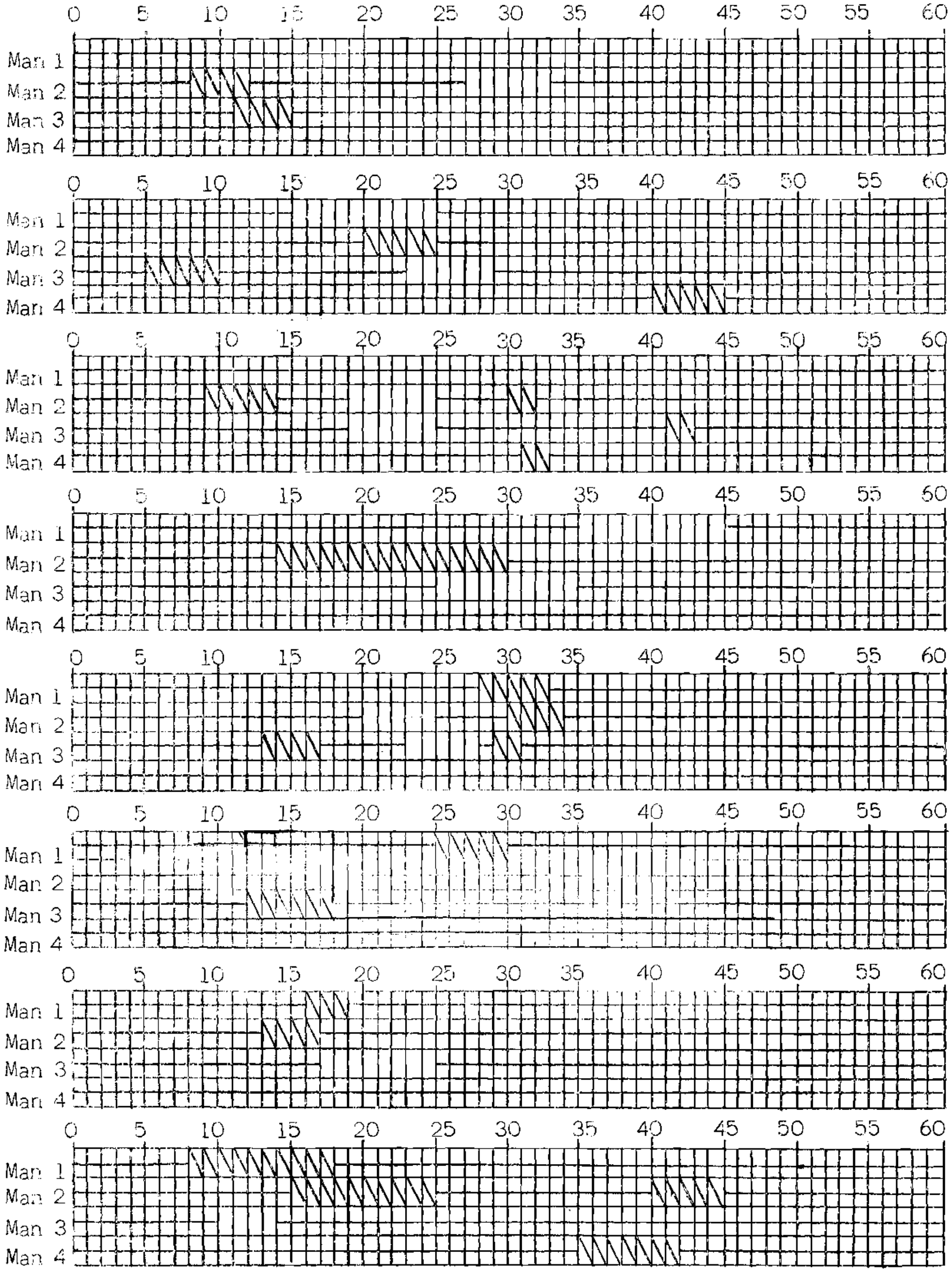
ACTIVITY ARRAY IN MINUTES FOR WEEK 2, DAY 2, HOURS 1 - 8



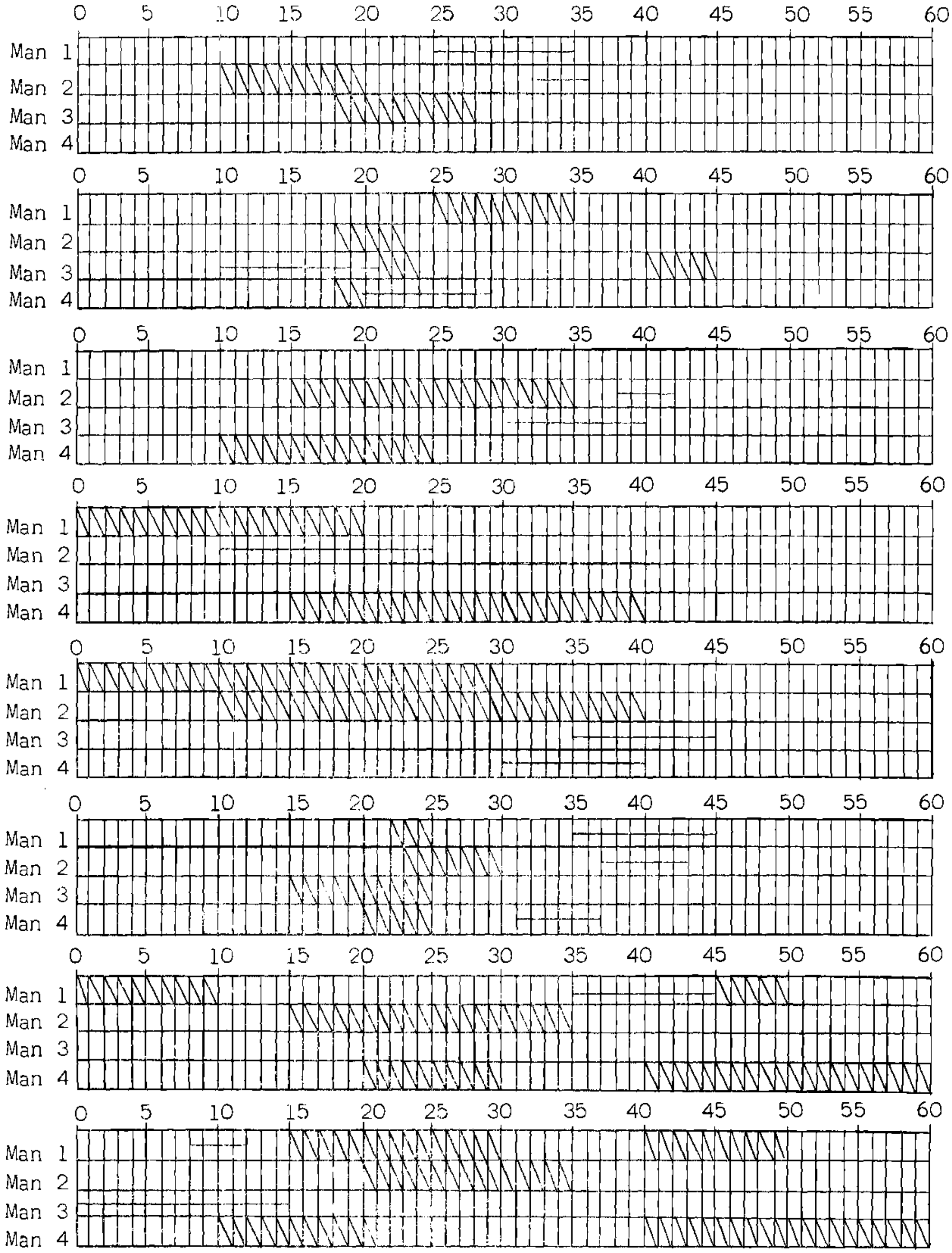
ACTIVITY ARRAY IN MINUTES FOR WEEK 2, DAY 3, HOURS 1 - 8



ACTIVITY ARRAY IN MINUTES FOR WEEK 2, DAY 4, HOURS 1 - 8



ACTIVITY ARRAY IN MINUTES FOR WEEK 2, DAY 5, HOURS 1 - 8



BIBLIOGRAPHY

1. Abruzzi, A., "Determining Reliable Allowances," The Management Review, Vol. 37, No. 4, April 1948.

2. Abruzzi, A., "Estimating and Evaluating Delay Factors," Time and Motion Study, 1, 12, 1952.

3. Abruzzi, A., "Delay Allowandes by Statistical Methods," Columbia Engineering Quarterly, May 1948.

4. Abruzzi, A., Work Measurement: New Principles and Procedures, Columbia University Press, New York, 1952.

5. Barlament, R. L., "Photographic Work Sampling (Memo-Motion) - New Cost Control Technique," Textile World, Vol. 108, September 1958.

6. Barnes, R. M., "Work Sampling," Proceedings 12th Management Engineers Conference, SAM-ASME, New York, April 1957.

7. Barnes, R. M., "Measuring Work by Sampling Methods," Proceedings 7th Conference of American Institute of Industrial Engineers, Washington, D. C., May 1956.

8. Barnes, R. M., Work Sampling, John Wiley & Sons, New York, 1957.

9. Barnes, R. M., and R. B. Andrews, "Work Sampling," Proceedings 8th Industrial Engineers Institute, University of California, February 1956.

10. Barnes, R. M., and R. B. Andrews, "Performance Sampling," UCLA, 1955.

11. Barnes, R. M., Buffa, E. S., Demangate, D. C., and R. B. Andrews, "Performance Sampling in Work Measurement," Proceedings 6th I. E. Institute, University of California, February 1954.

12. Barnes, R. M., Buffa, E. S., and J. A. Gary, "A Statistical Study of Performance Sampling in Work Measurement," UCLA, June 1953, Bulletin 510.

13. Barnes, R. M., and R. B. Andrews, "Performance Sampling in Work Measurement," The Journal of Industrial Engineering, Vol. VI, No. 6, Nov.-Dec. 1955.

14. Brisley, C. L., "How You Can Put Work Sampling to Work," Factory, Vol. 110, No. 7, July 1952.

15. Cochran, W. G., Sampling Techniques, John Wiley & Sons, New York, 1953.

16. Conway, R. W., "Some Statistical Considerations in Work Sampling," Journal of Industrial Engineering, Vol. III, No. 2, March-April 1957.

17. Correl, D. S., and R. M. Barnes, "Industrial Application of the Ratio-Delay Method," Advanced Management, Vol. XV, No. 8, August 1950, and Vol. XV, No. 9, September 1950.

18. Cote, L. J., and P. J. Scott, "Comparison of All-Day Time Study with Work Sampling by Use of Analysis of Variance," Journal of Industrial Engineering, Vol. 7, No. 1, Jan.-Feb. 1956.

19. Davidson, H. O., "Activity Sampling and Analysis - Present State of the Theory and Practice," American Society of Mechanical Engineering, Paper No. 53-F-24, 1953.

20. Deming, W. E., Some Theory of Sampling, John Wiley & Sons, New York, 1950.

21. Deming, W. E., "On Simplifications of Sampling Design Through Replication with Equal Probabilities and Without Stages," The Journal of the American Statistical Association, Vol. 51, March 1956, pp. 24-53.

22. Feller, William, An Introduction to Probability Theory and its Applications, Vol. I, John Wiley and Sons, New York, 1950.

23. Fisher, R. A., "The Arrangement of Field Experiments," Journal of the Ministry of Agriculture, 33, 1926, pp. 503-513; The Design of Experiments, London, Oliver and Boyd, 1935.

24. Fraser, D. A. S., Statistics - An Introduction, John Wiley and Sons, New York, 1958.

25. Gambrell, C.B. and J. W. Barany, "Automatic Work Sampling," Arizona State University, Tempe, Arizona, January 1959.

26. Gambrell, C. B., et al., "Less Expensive Work Sampling," Arizona State University, Tempe, Arizona, January 1956.

27. Givens, M. B., "An Experiment in Advisory Service," Journal of the Royal Statistical Society, 29, 1934, p. 394; "Government Statistics," Social Science Research Council, Bulletin 26, 1937, p. 132.

28. Halley, Edmund, "An Estimate of the Degrees of the Mortality of Mankind Drawn from Curious Tables of the Births and Funerals of the City of Breslau," Phil. Trans., 1693, pp. 596-610.

29. Halsey, J. J., "A New Model for Work Sampling - The GREDS Theory," Journal of Industrial Engineering, Nov.-Dec. 1960.

30. Hansen, B. L., Work Sampling, Prentice-Hall Inc., Englewood Cliffs, N. J., 1960.

31. Hansen, M. H., Hurwitz, and W. G. Madow, Sample Survey Methods and Theory, 2 Vols, John Wiley & Sons, New York, 1953.

32. Heiland, R., and W. Richardson, Work Sampling, McGraw-Hill Book Company, New York, 1957.

33. Lawes, J. B., and J. H. Gilbert, "On the Home Produce, Imports, Consumption, and Price of Wheat," Journal of the Royal Statistical Society, 43, 1880, p. 313f.

34. Malcolm, D. G., and L. L. Sammet, "Work Sampling Applications," Journal of Industrial Engineering, Vol. V, No. 3, 1954.

35. McAllister, G. E., "Random Ratio Delay," Journal of Industrial Engineering, Vol. IV, No. 3, 1953.

36. Mindlin, A., "The Application of Sample Survey Methodology to Work Sampling," Journal of Industrial Engineering, Vol. X, No. 4, July-Aug. 1959.

37. Moder, J. J., and W. J. Halladay, "Work Sampling Applied to Long Cycle Operations Performed by a Variable Labor Force," Journal of Industrial Engineering, Vol. VII, No. 4, July-Aug. 1956.

38. Moder, J. J., and D. P. Gaver, "Combination Attribute - Variable Activity Sampling," Unpublished Paper, 1962.

39. Morrow, R. L., "Ratio Delay Study," Mechanical Engineering, 63, 4, 1940.

40. Morrow, R. L., Time Study and Motion Economy, Ronald Press, New York, 1946.

41. Parzen, Emanuel, Modern Probability Theory and Its Applications, John Wiley and Sons, New York, 1960.

42. Rosander, A. C., "Work Sampling Techniques," Industrial Quality Control, Vol. XVI, No. 12, 1960.

43. Rosander, A. C., et al., "Use of Work Sampling for Cost Analysis and Control," American Statistical Association Journal, Vol. 53, June 1958.

44. Sammet, L. L., and D. G. Malcolm, "Work Sampling Studies: Guides to Analysis and Accuracy Criteria," Journal of Industrial Engineering, Vol. 5, No. 4, 1954.

45. Schaeffer, E. H., "Observation Ratios - A Short Cut to Time and Cost Analyses," Factory, Vol. 99, No. 7, 1941.

46. Shewhart, W. A., Economic Control of Quality of Manufactured Product, New York, D. Van Nostrand Co., 1931; T. C. Fry, Probability and Its Engineering Uses, D. Van Nostrand Co., 1928.

47. Stephan, F. F., "History of the Uses of Modern Sampling Procedures," Journal of the American Statistical Association, 43, 1948, pp. 12-39.

48. Tippett, L. H. C., "The Ratio Delay Technique," Time and Motion Study, Sawell Publications Ltd., London, E. C., May 1953.

49. Tippett, L. H. C., "Statistical Methods in Textile Research. Uses of the Binomial and Poisson Distribution. A Snap-Reading Method of Making Time Studies of Machines and Operatives in Factory Surveys," Journal of Textile Institute Transactions, Vol. 26, February 1935.

50. Tippett, L. H. C., Private correspondence with W. W. Hines, September 5, 1962.

51. Torgersen, P. E., "An Example of Work Sampling in the Hospital," Journal of Industrial Engineering, Vol. X, No. 3, May-June 1959.

52. Williams, J. A. C., "Work Sampling Techniques in Work Study," The Engineer, Vol. 203, February 1957.

VITA

Vernon Eugene McBryde was born February 3, 1933 at Palmyra, Lincoln County, Arkansas, the son of Robert Coy and Clara Mae (nee Carlton) McBryde. He was the second of three children. He spent his entire childhood in Arkansas, graduating from Glendale High School, Glendale, Arkansas, in May, 1950. He was valedictorian of a class of twenty graduates.

His college training began in September, 1950, at Arkansas State Teachers College in Conway, Arkansas. Interrupted by the Korean conflict after one term, he accumulated additional college credit at a number of universities during a four year enlistment in the United States Air Force. Among these were the University of Alabama, the University of Wichita, Kansas, and Trinity University of San Antonio, Texas. He was discharged in January, 1955 with a rank of Staff Sergeant after serving in various capacities of management analysis work. One year of his enlistment was spent in Korea.

He entered the University of Arkansas in January, 1955 as a sophomore in Industrial Engineering and received the degree BSIE with high honors in August 1957. During the summer of 1956 he was employed as a junior engineer by the General Electric Company at Louisville, Kentucky. Upon graduation he joined the Industrial Engineering staff of the Clary Corporation (now Remington-Rand) at Searcy, Arkansas.

In September, 1958, he joined the Industrial Engineering faculty at the University of Arkansas as an instructor. For the following two years he taught full time and did graduate work in Industrial Engineering. He was awarded the MSIE degree in June 1960. During the summer of 1959 he attended the National Science Foundation's First Summer Institute in Statistics for College Teachers at the University of Wyoming. The summer of 1960 was spent as an employee of the General Electric Company at the Pittsfield, Massachusetts,works.

He returned to the University of Arkansas in the Fall of 1960 as an Assistant Professor of Industrial Engineering. He attended the National Science Foundation's Second Summer Institute in Statistics held in Iowa State University during the summer of 1961. He also participated in the Second Institute on Effective Teaching at the Pennsylvania State University in August-September, 1961.

He commenced work toward the doctorate degree at the Georgia Institute of Technology in September 1961 as a Ford Foundation Fellow. Upon finishing this degree he returned to the University of Arkansas as an Associate Professor in Industrial Engineering.

The author is married to the former Miss Nellie Florence Sims of Harrison, Arkansas. They have one son, Carlton Bryan, born June 4, 1957, and one daughter, Pamella Jayne, born November 8, 1960.